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Inferences for Extended Generalized Exponential Distribution Based on Order Statistics

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Abstract. Recently, a new distribution, named as extended generalized exponential distribution, has been introduced by Kundu and Gupta (2011). In this paper, we consider the extended generalized exponential distribution with known shape parameters α and β. At first, the exact expressions for marginal and product moments of order statistics are derived. Then, these values are used to obtain the necessary coefficients for the best linear unbiased estimators and L-moments estimators of the location and scale parameters. The mean squared errors of these estimators are also given and compared.

Keywords. Best linear unbiased estimators, Hypergeometric function, L-moments estimators, Location parameter, Moments of order statistics, Scale parameter.

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1 Introduction

The two-parameter generalized exponential (GE) distribution has a probability density function (pdf) and a cumulative distribution function (cdf) of the forms

$$\begin{aligned} f(x;\alpha,\lambda) &= \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}, \ \ x>0, \\ F(x;\alpha,\lambda) &= (1-e^{-\lambda x})^{\alpha}, \end{aligned}$$

where $\alpha > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter. When $\alpha = 1$, the GE distribution corresponds to the exponential distribution and, when α is a positive integer, the GE cdf is the cdf of the maximum of a random sample of size α from the standard exponential distribution.

The GE distribution has been proposed by Gupta and Kundu (1999) as an alternative to gamma and Weibull distributions. The hazard rate of this distribution is an increasing or a decreasing function depending on the shape parameter. Gupta and Kundu (1999) have shown that the three-parameter GE model (shape, location and scale parameters) fits better than the three-parameter gamma or three-parameter Weibull in some cases.

The GE distribution has been studied extensively by Gupta and Kundu (2001a,b), Raqab (2005), Raqab and Madi (2005), Alamm *et al.* (2007), Gupta and Kundu (2007), Baklizi (2008), Kundu and Gupta (2008), Mitra and Kundu (2008), Madi and Raqab (2009), Wong and Wu (2009) and Chen and Lio (2010). Moreover, two discrete analogues of the generalized exponential distribution have been introduced and studied by Nekoukhou *et al.* (2012, 2013).

One of the major disadvantages of the GE distribution is that it cannot be used to analyze a data set with non-monotone hazard functions, similar to Weibull or gamma distributions.

Recently, Kundu and Gupta (2011) introduced a new family of distribution functions as an extension of the GE distribution with an additional shape parameter. The four-parameter extended generalized exponential (EGE) distribution has the distribution function of the form

$$F(x;\alpha,\beta,\mu,\lambda) = \begin{cases} \left(1-(1-\beta\lambda(x-\mu))^{\frac{1}{\beta}}\right)^{\alpha} & \beta \neq 0 \\ \\ \left(1-e^{-\lambda(x-\mu)}\right)^{\alpha} & \beta = 0, \end{cases} \tag{1.1}$$

where $\alpha > 0$ and $-\infty < \beta < \infty$ are the shape parameters, $-\infty < \mu < \infty$ is the location parameter and $\lambda > 0$ is the scale parameter.

The pdf of EGE distribution becomes

$$f(x;\alpha,\beta,\mu,\lambda) = \begin{cases} \alpha\lambda\left(1-(1-\beta\lambda(x-\mu))^{\frac{1}{\beta}}\right)^{\alpha-1}(1-\beta\lambda(x-\mu))^{\frac{1}{\beta}-1} & \beta \neq 0 \\ \\ \alpha\lambda\left(1-e^{-\lambda(x-\mu)}\right)^{\alpha-1}e^{-\lambda(x-\mu)} & \beta = 0, \end{cases} \tag{1.2}$$

when $\mu < x < \infty$ for $\beta \leq 0$ and $\mu < x < \mu + 1/(\beta\lambda)$ for $\beta > 0$.

Many well known distributions can be obtained as special cases of the EGE distribution. If $\beta = 0$, EGE reduces to GE; $\beta = 0,\ \alpha = 1$, EGE reduces to exponential; $\beta = 1,\ \alpha = 1$, EGE reduces to uniform; $\alpha = 1$, EGE reduces to generalized Pareto and $\beta < 0,\ \alpha = 1$, EGE reduces to Pareto.

The hazard function of EGE distribution is unimodal if $\beta < 0,\ \alpha > 1$, a decreasing function if $\beta < 0,\ \alpha < 1$, an increasing function if $\beta > 0,\ \alpha > 1$, and bathtub shaped if $\beta > 0,\ \alpha < 1$.

For $\lambda = 1$, $\mu = 0$, $\beta \neq 0$ and $k\beta + 1 > 0$,

$$E[(1-\beta X)^k] = \frac{\Gamma(\alpha+1)\Gamma(k\beta+1)}{\Gamma(\alpha+k\beta+1)}.$$

Suppose that $X_1, ..., X_n$ are independent and identically distributed (iid) observations from the cdf F and the pdf f. The order statistics of the sample is defined by the arrangement of $X_1, ..., X_n$ from the smallest to the largest, denoted as $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$. Order statistics have been used in a wide range of problems, including characterization of probability distributions, goodness of fit tests, robust statistical estimation, reliability analysis, analysis of censored samples and entropy estimation.

Order statistics from the ordinary exponential distribution were studied by Joshi (1982). Raqab and Ahsanullah (2001) derived exact expressions for the marginal and product moments of order statistics from the GE distribution ($\beta = 0$). Further, they used the moments to obtain estimators of the location and scale parameters of this model. For more details on order statistics from generalized exponential distributions, one can refer to Raqab (2004) and Sultan (2007).

In this paper, we derive exact expressions for means, variances and covariances of order statistics from EGE distribution. Without loss of generality, we take $\mu = 0$ and $\lambda = 1$ when computing the moments. As an application, we use these moments to compute the coefficients required for the best linear unbiased estimators (BLUE's) and L-moments estimators (LME's) of the location and scale parameters.

2 Moments of Order Statistics

Let $X_1, ..., X_n$ be a random sample of EGE$(\alpha, \beta, 0, 1)$. The pdf of $X_{r:n}$ is given by (Arnold *et al.*, 1992)

$$f_{r:n}(x; \alpha, \beta) = nc_{r-1}^{n-1}[F(x)]^{r-1}[1 - F(x)]^{n-r} f(x), \tag{2.1}$$

where $c_r^n = \frac{n!}{r!(n-r)!}$.

Theorem 2.1. *Let $X_{r:n}$ be the r-th order statistic of the EGE distribution with $\mu = 0$, $\lambda = 1$ and $\beta \neq 0$. Then for $k\beta + 1 > 0$*

$$E\left[(1 - \beta X_{r:n})^k\right] = \alpha nc_{r-1}^{n-1} \sum_{i=0}^{n-r} (-1)^i c_i^{n-r} B(\alpha(r + i), k\beta + 1), \tag{2.2}$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

Proof. From (1.1), (1.2), (2.1) and using $u = (1 - \beta x)^{\frac{1}{\beta}}$, we can write

$$\begin{aligned} E\left[(1 - \beta X_{r:n})^k\right] &= \alpha nc_{r-1}^{n-1} \int_0^1 u^{k\beta}(1 - u)^{\alpha r-1}[1 - (1 - u)^{\alpha}]^{n-r} du \\ &= \alpha nc_{r-1}^{n-1} \int_0^1 u^{k\beta}(1 - u)^{\alpha r-1} \sum_{i=0}^{n-r} (-1)^i c_i^{n-r} (1 - u)^{\alpha i} du \\ &= \alpha nc_{r-1}^{n-1} \sum_{i=0}^{n-r} (-1)^i c_i^{n-r} B(\alpha(r + i), k\beta + 1). \end{aligned}$$

□

Corollary 2.1. From (2.2), it easily follows that

$$\theta_{r:n} = E(X_{r:n}) = \frac{1}{\beta}\left[1 - \alpha nc_{r-1}^{n-1} \sum_{i=0}^{n-r} (-1)^i c_i^{n-r} B(\alpha(r + i), \beta + 1)\right], \tag{2.3}$$

$$\begin{aligned} Var(X_{r:n}) &= \frac{\alpha nc_{r-1}^{n-1}}{\beta^2} \sum_{i=0}^{n-r} (-1)^i c_i^{n-r} B(\alpha(r + i), 2\beta + 1) \\ &- \frac{1}{\beta^2}\left[\alpha nc_{r-1}^{n-1} \sum_{i=0}^{n-r} (-1)^i c_i^{n-r} B(\alpha(r + i), \beta + 1)\right]^2. \end{aligned} \tag{2.4}$$

Remark 1. Note that when $n = 1$, then the results of Theorem 2.1 coincide with the mean and variance of parent random variable X which is given by Kundu and Gupta (2011). Also for $\beta = 0$, the results are available in Raqab and Ahsanullah (2001).

To derive the corresponding moment generation function (MGF) of $X_{r:n}$, we can write the pdf of $X_{r:n}$ (David, 1981) as

$$f_{r:n}(x;\alpha,\beta) = \sum_{i=0}^{n-r} d_i(n,r) f(x;\alpha(r+i),\beta), \tag{2.5}$$

where $d_i(n,r) = \frac{(-1)^i n c_{r-1}^{n-1} c_i^{n-r}}{r+i}$.

Further, the MGF of $1 - \beta X$ is given by

$$M_{1-\beta X}(t) = \int_0^{\frac{1}{\beta}} e^{t(1-\beta x)} \alpha [1 - (1-\beta x)^{\frac{1}{\beta}}]^{\alpha-1} (1-\beta x)^{\frac{1}{\beta}-1} dx. \tag{2.6}$$

Making substitution $u = (1-\beta x)^{\frac{1}{\beta}}$, (2.6) reduces to

$$M_{1-\beta X}(t) = \alpha \int_0^1 e^{tu^{\beta}} (1-u)^{\alpha-1} du.$$

By using the Maclaurin series expansion $e^{tu^{\beta}} = \sum_{j=0}^{\infty} \frac{(tu^{\beta})^j}{j!}$ and integrating the results, we obtain

$$M_{1-\beta X}(t) = \alpha \sum_{j=0}^{\infty} B(\alpha, j\beta + 1) \frac{t^j}{j!}. \tag{2.7}$$

Now, from (2.5) and (2.7), the MGF of $X_{r:n}$ is derived to be

$$M_{X_{r:n}}(t) = \alpha e^{\frac{t}{\beta}} \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} (-1)^{i+j} n c_{r-1}^{n-1} c_i^{n-r} B(\alpha(r+i), j\beta + 1) \frac{t^j}{\beta^j j!}.$$

The mean and variance of the r-th order statistic given in (2.3) and (2.4) can be obtained similarly by differentiating $M_{X_{r:n}}(t)$ and evaluating at $t = 0$.

The joint pdf of any two order statistics $X_{r:n}$ and $X_{s:n}$, $1 \leq r < s \leq n$, can be written as (David, 1981)

$$\begin{aligned} f_{r,s:n}(x,y) =& c_{r,s:n}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y) \\ =& c_{r,s:n}\sum_{i=0}^{n-s}\sum_{j=0}^{s-r-1}(-1)^{i+j}c_i^{n-s}c_j^{s-r-1}[F(x)]^{r+j-1}[F(y)]^{s-r+i-j-1}f(x)f(y) \\ =& \alpha^2 c_{r,s:n}\sum_{i=0}^{n-s}\sum_{j=0}^{s-r-1}\{(-1)^{i+j}c_i^{n-s}c_j^{s-r-1}[1-(1-\beta x)^{\frac{1}{\beta}}]^{\alpha(r+j)-1}(1-\beta x)^{\frac{1}{\beta}-1} \\ & \times [1-(1-\beta y)^{\frac{1}{\beta}}]^{\alpha(s-r+i-j)-1}(1-\beta y)^{\frac{1}{\beta}-1}\}, \end{aligned} \quad (2.8)$$

where $c_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Using (2.8) and transformations $v = (1-\beta x)^{\frac{1}{\beta}}$ and $u = (1-\beta y)^{\frac{1}{\beta}}$, we get

$$\begin{aligned} E[(1-\beta X_{r:n})(1-\beta X_{s:n})] &= \alpha^2 c_{r,s:n}\sum_{i=0}^{n-s}\sum_{j=0}^{s-r-1}\{(-1)^{i+j}c_i^{n-s}c_j^{s-r-1} \\ &\quad \times \int_0^1 v^{\beta}(1-v)^{\alpha(r+j)-1}B_v(\beta+1,\alpha(s-r+i-j))dv\}, \end{aligned}$$

where $B_v(a,b) = \int_0^v z^{a-1}(1-z)^{b-1}dz,\ 0 \leq v \leq 1$ is incomplete beta function.

Now, we need the generalized hypergeometric function

$${}_pF_q(\gamma_1,...,\gamma_p;\eta_1,...,\eta_q;u) = \sum_{i=0}^{\infty}\frac{(\gamma_1)_i...(\gamma_p)_i}{(\eta_1)_i...(\eta_q)_i}\frac{u^i}{i!},$$

where $(\gamma)_i = \gamma(\gamma+1)...(\gamma+i-1) = \Gamma(\gamma+i)/\Gamma(\gamma),\ \gamma \neq 0,\ i = 1,2,...,$ and p and q are nonnegative integers. It has been shown by Mathai (1993) that

$$B_v(a,b) = \frac{1}{a}v^a\ {}_2F_1(a,1-b;a+1;v),$$

and

$$\int_0^1 t^{\rho-1}(1-t)^{\tau-1}\ {}_2F_1(c,d;\xi;t)dt = B(\rho,\tau){}_3F_2(c,d,\rho;\xi,\rho+\tau;1),$$

for $\rho, \tau > 0$ and $\xi + \tau - c - d > 0$. Hence, we obtain

$$\begin{aligned} E[(1-\beta X_{r:n})(1-\beta X_{s:n})] &= \alpha^2 c_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} \frac{(-1)^{i+j} c_i^{n-s} c_j^{s-r-1}}{\beta+1} B(2\beta+2, \alpha(r+j)) \\ & \quad {}_3F_2(\beta+1, 1-\alpha(s-r+i-j), 2\beta+2; \beta+2, \\ & \quad 2\beta+2+\alpha(r+j); 1), \end{aligned}$$

After some simplification, the expressions for product moments $\theta_{r,s:n} = E(X_{r:n}X_{s:n})$, $1 \leq r < s \leq n$ are achieved (for the case $r = s$, $\theta_{r,r:n} = \theta^{(2)}_{r:n} = E(X^2_{r:n})$, are obtained from (2.2)). These values are used to evaluate the covariances $\eta_{r,s:n} = Cov(X_{r:n}, X_{s:n}) = \theta_{r,s:n} - \theta_{r:n}\theta_{s:n}$. The values of means, variances and covariances were computed by using R. The means of order statistics for some values of n, α and β are presented in Table 1. Other values are not presented here, but they can be given upon request from the author.

Using similar calculations, we get the joint MGF of $X_{r:n}$ and $X_{s:n}$ as

$$\begin{aligned} M_{r,s:n}(t_1, t_2) &= e^{\frac{t_1+t_2}{\beta}} \alpha^2 c_{r,s:n} \sum_{i=0}^{n-s} \sum_{j=0}^{s-r-1} (-1)^{i+j} c_i^{n-s} c_j^{s-r-1} \\ & \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{(k+m)}}{(k\beta+1)\beta^{(k+m)}} B[(k+m)\beta+2, \alpha(r+j)] \\ & \quad {}_3F_2(k\beta+1, 1-\alpha(s-r+i-j), (m+k)\beta+2; k\beta+2, \\ & \quad (m+k)\beta+2+\alpha(r+j); 1) \frac{t_1^m}{m!} \frac{t_2^k}{k!}. \end{aligned}$$

3 Location and Scale Parameters Estimation

As an application of the moments of order statistics obtained in Section 2, we are going to use them for estimating the location parameter μ and scale parameter $\sigma = 1/\lambda$. Toward this end, we will apply two methods which are based on linear combinations of order Statistics.

3.1 Best Linear Unbiased Estimators

David (1981) has proposed a method of finding the best linear unbiased estimators (BLUE's) of parameters in location-scale families based on linear combinations of order statistics. Let $Y_{1:n}, ..., Y_{n:n}$ be an ordered random sample of size n from a location-scale family with location parameter μ and scale parameter σ. Then, the BLUE's of μ and σ are as

$$\hat{\mu}_B = C_{11}Y_{1:n} + C_{12}Y_{2:n} + ... + C_{1n}Y_{n:n},$$

$$\hat{\sigma}_B = C_{21}Y_{1:n} + C_{22}Y_{2:n} + ... + C_{2n}Y_{n:n},$$

where

$$C = (A^T V^{-1} A)^{-1} A^T V^{-1},$$

$A = (\underline{1}\ \underline{\theta})$, $\underline{1}^T = (1, ..., 1)_{1\times n}$, $\underline{\theta}^T = (\theta_{1:n}, ..., \theta_{n;n})_{1\times n}$ and V^{-1} is the inverse of the covariance matrix $V = (\eta_{r,s:n})_{n\times n}$. The coefficient matrix C can be easily obtained based on the computations of the expected values, variances and covariances of order statistics from the standardized distribution ($\mu = 0,\ \sigma = 1$).

Now, suppose that $Y_{1:n}, ..., Y_{n:n}$ are an ordered random sample of size n from the EGE distribution with the pdf given in (1.2), where the shape parameters α and β are known. Therefore, in this case, $Y_{1:n}, ..., Y_{n:n}$ can be considered as a random sample of a location-scale family. Moreover, $\underline{X} = (X_{1:n}, ..., X_{n:n})$ where $X_{i:n} = \frac{Y_{i:n}-\mu}{\sigma}$, $i = 1, ..., n$, $\sigma = 1/\lambda$ is an ordered sample from a population with the standard EGE distribution (EGE$(\alpha, \beta, 0, 1)$). Table 2 provides the coefficients of the BLUE's of the location and scale parameters, for some values of sample size n and shape parameters α and β.

Variances and covariances of these estimators are given by

$$Var(\hat{\mu}_B) = d_{11}\sigma^2, \quad Var(\hat{\sigma}_B) = d_{22}\sigma^2, \quad Cov(\hat{\mu}_B, \hat{\sigma}_B) = d_{12}\sigma^2,$$

where

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix} \sigma^2 = (A^T V^{-1} A)^{-1}.$$

3.2 L-moment Estimators

Here, we propose a method of estimating μ and σ based on another linear combination of order statistics. The estimators obtained by this method are popularly known as L-moment estimators (Hosking, 1990). The first and second sample L-moments are as

$$L_1 = \frac{1}{n}\sum_{i=1}^{n} Y_{i:n} = \bar{x}, \quad L_2 = \frac{1}{n(n-1)}\sum_{i=1}^{n}(2i - n - 1)Y_{i:n},$$

and first two population L-moments are

$$\begin{aligned}\lambda_1 &= E(Y) = \sigma E(X) + \mu = \frac{\sigma}{\beta}(1 - h_0) + \mu,\\ \lambda_2 &= E(Y_{2:2} - Y_{1:2})/2 = \frac{\sigma}{\beta}(h_0 - h_1),\end{aligned}$$

respectively, where $h_0 = \alpha B(\alpha, \beta + 1)$ and $h_1 = 2\alpha B(2\alpha, \beta + 1)$.

Now, to obtain the L-moment estimators (LME's) of the unknown parameters μ and σ, we need to equate the sample L-moments with the population L-moments. Therefore, $\hat{\mu}_L$ and $\hat{\sigma}_L$ are obtained from

$$\begin{cases} \frac{\sigma}{\beta}(1 - h_0) + \mu = \frac{1}{n}\sum_{i=1}^{n} Y_{i:n} \\ \frac{\sigma}{\beta}(h_0 - h_1) = \frac{1}{n(n-1)}\sum_{i=1}^{n}(2i - n - 1)Y_{i:n}, \end{cases}$$

as

$$\begin{aligned}\hat{\mu}_L &= \sum_{i=1}^{n} \frac{(n-1)(h_0 - h_1) - (2i - n - 1)(1 - h_0)}{n(n-1)(h_0 - h_1)} Y_{i:n} = \sum_{i=1}^{n} a_i Y_{i:n}\\ \hat{\sigma}_L &= \sum_{i=1}^{n} \frac{\beta(2i - n - 1)}{n(n-1)(h_0 - h_1)} Y_{i:n} = \sum_{i=1}^{n} b_i Y_{i:n},\end{aligned}$$

where $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} b_i = 0$.

Mean square errors (MSE's) of these estimators are given by

$$MSE(\hat{\mu}_L) = \sigma^2 E[(\sum_{i=1}^{n} a_i X_{i:n})^2]$$

and

$$MSE(\hat{\sigma}_L) = \sigma^2 E[(\sum_{i=1}^{n} b_i X_{i:n} - 1)^2].$$

Table 3, provides and compares the MSE's (variances) of BLUE's of μ and σ and the MSE's of LME's of μ and σ in terms of σ^2. From Table 3, it follows that LME's have larger MSE's than BLUE's. However, their differences is small for $\beta > 0$, specially when σ is small. On the other hand, LME's have easier computations than BLUE's.

Examples 3.1. We assume that the failure time of a component follows the EGE distribution with $\alpha = 1.5$ and $\beta = -0.3$. Suppose that we have the simulated failure times from

EGE(μ, σ) as 5.04696, 5.09444, 5.1042, 5.4557, 5.7921, 5.8528, 6.0994, 6.3989, 6.7436, 6.8353. Using the coefficients proposed in Table 2, the BLUE's of μ and σ are computed to be $\hat{\mu}_B = 4.9086$ and $\hat{\sigma}_B = 0.5869$. The corresponding variances and covariances of $\hat{\mu}_B$ and $\hat{\sigma}_B$ are as

$$Var(\hat{\mu}_B) = 0.03863\sigma^2, \quad Var(\hat{\sigma}_B) = 0.15783\sigma^2$$

and

$$Cov(\hat{\mu}_B, \hat{\sigma}_B) = -0.03703\sigma^2.$$

Moreover, the LME's of μ and σ can be computed as $\hat{\mu}_L = 5.0885$ and $\hat{\sigma}_L = 0.39944$.

To estimate the population mean

$$\eta = \frac{1 - \frac{\Gamma(\alpha+1)\Gamma(k\beta+1)}{\Gamma(\alpha+k\beta+1)}}{\beta}\sigma + \mu,$$

we can use $\tilde{\eta} = \bar{x}$, the sample mean, where it is computed to be $\tilde{\eta} = 5.8423$. The standard error $S.E.(\tilde{\eta}) = 0.2124$. The BLUE of η is

$$\hat{\eta} = \frac{1 - \frac{\Gamma(\alpha+1)\Gamma(k\beta+1)}{\Gamma(\alpha+k\beta+1)}}{\beta}\hat{\sigma} + \hat{\mu} = 6.0163.$$

The standard error of $\hat{\eta}$ is computed to be $S.E.(\hat{\eta}) = 0.1587$. Therefore, we observe that the BLUE's perform better than the sample mean in the sense of standard errors.

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Table 1: Means of order statistics

β	n	r	$\alpha = 0.5$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 3$
-0.3	3	1	0.13034	0.60564	0.82088	1.19555
		2	0.51157	1.43702	1.78344	2.34994
		3	1.88707	3.61856	4.20241	5.12864
	5	1	0.05440	0.39173	0.56524	0.87945
		2	0.18205	0.79229	1.04990	1.48583
		3	0.43081	1.32913	1.66755	2.22126
		4	0.94874	1.22558	2.67147	3.38561
		5	2.59894	4.69661	5.39035	6.48472
	10	1	0.01587	0.22657	0.35781	0.61059
		2	0.04925	0.41730	0.60482	0.93991
		3	0.10301	0.61521	0.84628	1.24373
		4	0.18198	0.83856	1.10682	1.56153
		5	0.29426	1.09462	1.40577	1.91872
		6	0.45141	1.41400	1.76891	2.34604
		7	0.68922	1.83451	2.24116	2.89516
		8	1.06154	2.44468	2.91956	3.67637
		9	1.75498	3.50323	4.08657	5.00952
		10	3.82562	6.48472	7.35114	8.71216
0.5	3	1	0.10951	0.43543	0.55611	0.73555
		2	0.35619	0.80339	0.92587	1.08913
		3	0.82190	1.22687	1.31801	1.43245
	5	1	0.04996	0.31408	0.42866	0.60805
		2	0.15560	0.55769	0.68863	0.87235
		3	0.32854	0.79681	0.92318	1.09008
		4	0.59365	1.05786	1.16669	1.30465
		5	1.01825	1.38305	1.45947	1.55341
	10	1	0.01541	0.19880	0.29921	0.46947
		2	0.04683	0.34328	0.46788	0.65708
		3	0.09520	0.47380	0.60746	0.79836
		4	0.16199	0.59987	0.73540	0.92104
		5	0.24943	0.72626	0.85900	1.03516
		6	0.36088	0.85667	0.98294	1.14634
		7	0.50172	0.99513	1.11153	1.25901
		8	0.68127	1.14745	1.25026	1.37821
		9	0.91808	1.32431	1.40868	1.51204
		10	1.26118	1.55341	1.61090	1.68037
1.5	3	1	0.09221	0.31795	0.38570	0.47127
		2	0.25936	0.49075	0.53595	0.58432
		3	0.47031	0.60223	0.62120	0.63964
	5	1	0.04552	0.24957	0.32307	0.42150
		2	0.13215	0.39445	0.45821	0.53091
		3	0.25257	0.49873	0.54396	0.59095
		4	0.39563	0.57643	0.60301	0.62888
		5	0.54394	0.63239	0.64316	0.65313
	10	1	0.01489	0.17168	0.24574	0.35467
		2	0.04416	0.27710	0.35525	0.45376
		3	0.08699	0.35844	0.42998	0.51244
		4	0.14226	0.42528	0.48686	0.55349
		5	0.20845	0.48161	0.53218	0.58431
		6	0.28364	0.52951	0.56906	0.60823
		7	0.36532	0.57017	0.59924	0.62707
		8	0.45021	0.60424	0.62374	0.64189
		9	0.53388	0.63196	0.64315	0.65329
		10	0.60983	0.65313	0.65761	0.66159

Table 2: Coefficients for the BLUE's of μ and σ

			$\alpha = 0.5$		$\alpha = 1.5$		$\alpha = 2$		$\alpha = 3$	
β	n	r	μ	σ	μ	σ	μ	σ	μ	σ
-0.3	3	1	1.17796	-1.49546	1.49011	-0.82280	1.57789	-0.72016	1.70737	-0.60757
		2	-0.13253	1.18293	-0.39927	0.67797	-0.46850	0.59333	-0.57098	0.50010
		3	-0.04543	0.31252	-0.09084	0.14482	-0.10939	0.12683	-0.13638	0.10746
	5	1	0.77372	-1.09722	1.32404	-0.91097	1.38538	-0.77678	1.48197	-0.63612
		2	0.64114	-0.63153	-0.10181	0.35362	-0.10399	0.29176	-0.11144	0.22805
		3	-0.44070	1.21667	-0.11543	0.30316	-0.14368	0.26388	-0.18671	0.22143
		4	0.02187	0.41231	-0.08782	0.21046	-0.11326	0.18363	-0.15133	0.15538
		5	0.00396	0.09977	-0.01897	0.04371	-0.02433	0.03749	-0.03248	0.03125
	10	1	-0.15911	0.24438	1.12928	-0.91153	1.13829	-0.74447	1.16792	-0.57877
		2	0.11228	-0.14356	0.02728	0.12678	0.06382	0.08057	0.10988	0.03909
		3	1.60805	-2.33556	-0.01145	0.14769	-0.00394	0.11711	0.00449	0.08629
		4	-0.27433	0.66173	-0.02460	0.14486	-0.02906	0.12189	-0.03704	0.09764
		5	-0.24165	0.64917	-0.02852	0.13343	-0.03807	0.11492	-0.05334	0.09522
		6	-0.00721	0.27013	-0.02814	0.11774	-0.03935	0.10228	-0.05699	0.08603
		7	-0.01676	0.25358	-0.02533	0.09919	-0.03620	0.08618	-0.05322	0.07283
		8	-0.01164	0.20958	-0.02080	0.07791	-0.02999	0.06726	-0.04431	0.05666
		9	-0.00786	0.15614	-0.01460	0.05291	-0.02103	0.04504	-0.03094	0.03748
		10	-0.00173	0.03619	-0.00310	0.01099	-0.00442	0.00918	-0.00643	0.00749
0.5	3	1	1.13004	-1.48753	1.54681	-1.32647	1.69215	-1.34118	1.95617	-1.41429
		2	0.03622	0.12819	0.00630	0.11764	0.07333	0.05573	0.20152	-0.04187
		3	-0.16626	1.35933	-0.55312	1.20883	-0.76548	1.28545	-1.15770	1.45616
	5	1	0.82998	-0.94488	1.24146	-0.99731	1.28939	-0.96758	1.39168	-0.96418
		2	0.64362	-0.69237	0.06933	-0.00856	0.14648	-0.06709	0.27973	-0.15558
		3	-0.49950	0.66519	0.01288	0.05624	0.04668	0.02786	0.11259	-0.01914
		4	0.00915	0.12576	-0.02709	0.12358	-0.02595	0.11672	-0.01971	0.10584
		5	0.01674	0.84629	-0.29658	0.82604	-0.45660	0.89008	-0.76429	1.03307
	10	1	-0.18438	0.18208	1.02818	-0.77195	0.99026	-0.70260	0.96272	-0.63890
		2	0.05306	-0.04844	0.07792	-0.04293	0.14352	-0.08803	0.24155	-0.14835
		3	1.54311	-1.50561	0.04011	-0.01146	0.08207	-0.04079	0.15315	-0.08582
		4	-0.13309	0.15997	0.02227	0.00531	0.05052	-0.01500	0.10275	-0.04859
		5	-0.19104	0.22518	0.01160	0.01752	0.03034	0.00358	0.06758	-0.02090
		6	-0.01213	0.05473	0.00390	0.02892	0.01511	0.01997	0.03909	0.00352
		7	-0.01639	0.06772	-0.00279	0.04218	-0.00155	0.03760	-0.01242	0.02896
		8	-0.01117	0.07831	-0.01016	0.06113	-0.01320	0.06097	-0.01734	0.06098
		9	-0.00980	0.10990	-0.02119	0.09578	-0.03456	0.10120	-0.06030	0.11303
		10	-0.03815	0.67613	-0.14986	0.57548	-0.26564	0.62326	-0.50164	0.73608
1.5	3	1	1.26227	-2.61917	2.01119	-3.30901	2.41809	-3.86719	3.30682	-5.14955
		2	-0.03292	-0.04596	0.27343	-0.53184	0.60698	-1.04735	1.49834	-2.40416
		3	-0.22934	2.66514	-1.28463	3.84086	-2.02508	4.91454	-3.30517	7.55371
	5	1	0.99853	-1.76662	1.49864	-2.34095	1.68828	-2.60279	2.11371	-3.22038
		2	0.25299	-0.47113	0.15133	-0.25010	0.33605	-0.52986	0.78490	-1.20566
		3	-0.19522	0.29940	0.11595	-0.21235	0.28175	-0.46367	0.74402	-1.16088
		4	-0.00700	-0.08566	0.12822	-0.28498	0.31515	-0.56949	0.88050	-1.42432
		5	-0.04929	2.02402	-0.89416	3.08839	-1.62124	4.16583	-3.52315	7.01127
	10	1	-2.27322	3.60717	1.16731	-1.77208	1.20346	-1.81961	1.33087	-2.00516
		2	-1.97967	3.13985	0.10662	-0.16318	0.21821	-0.33094	0.45006	-0.67882
		3	6.42246	-10.19650	0.06859	-0.10673	0.15861	-0.24152	0.37417	-0.56514
		4	-0.05362	0.07850	0.05198	-0.08212	0.13048	-0.20009	0.33954	-0.51401
		5	-0.44407	0.69555	0.04320	-0.07071	0.11593	-0.17997	0.32550	-0.49468
		6	-0.12216	0.18087	0.03892	-0.06774	0.10974	-0.17408	0.32702	-0.50038
		7	-0.08764	0.11841	0.03839	-0.07409	0.11145	-0.18379	0.34692	-0.53748
		8	-0.05577	0.05075	0.04338	-0.09867	0.12590	-0.22271	0.40095	-0.63608
		9	-0.03037	-0.03902	0.06360	-0.18111	0.17604	-0.35096	0.55406	-0.92000
		10	-0.37591	2.36448	-0.62239	2.61646	-1.34985	3.70372	-3.44914	6.85180

Table 3: MSE's of the BLUE's and LME's of μ and σ in terms of σ^2

		$\alpha = 0.5$		$\alpha = 1.5$		$\alpha = 2$		$\alpha = 3$	
β	n	BLUE	LME	BLUE	LME	BLUE	LME	BLUE	LME
-0.3	3	0.07235	0.16701	0.58506	1.24582	0.93992	1.93962	1.72744	3.45206
		1.53621	2.14654	0.92325	1.48914	0.86782	1.41521	0.81700	1.34444
	5	0.02782	0.06566	0.15920	0.64517	0.27445	1.01702	0.54042	1.83484
		0.67944	1.25131	0.39577	0.86475	0.36765	0.82062	0.34310	0.77839
	10	0.00600	0.01056	0.03863	0.29416	0.07413	0.46775	0.16067	0.85175
		0.34018	0.60934	0.15783	0.42439	0.14477	0.40239	0.13406	0.38134
0.5	3	0.03144	0.03682	0.17359	0.17980	0.23685	0.24044	0.33740	0.33817
		0.37745	0.37845	0.22710	0.22790	0.22483	0.22500	0.23038	0.23046
	5	0.00666	0.00854	0.07174	0.08332	0.10711	0.11317	0.15588	0.16173
		0.16356	0.17332	0.09449	0.10038	0.09447	0.09931	0.09937	0.10335
	10	0.00782	0.00551	0.02381	0.03483	0.03837	0.04795	0.06224	0.06943
		0.08058	0.05829	0.03292	0.04008	0.03350	0.03974	0.03679	0.04197
1.5	3	0.01932	0.02163	0.09986	0.09989	0.13418	0.13460	0.18714	0.19100
		0.21056	0.21060	0.28131	0.28364	0.34686	0.35299	0.45307	0.46951
	5	0.00485	0.00871	0.04834	0.05113	0.06855	0.07057	0.09939	0.10277
		0.06734	0.07581	0.12209	0.13041	0.16458	0.17326	0.23090	0.24467
	10	0.02761	0.00231	0.01869	0.02312	0.02925	0.03245	0.04554	0.04810
		0.08194	0.01857	0.04386	0.05416	0.06721	0.07592	0.10346	0.11204