JIRSS (2015)
Vol. 14, No. 2, pp 37-52
DOI:10.7508/jirss.2015.02.003

Characterizations of Multivariate Normal-Poisson Model

Khoirin Nisa[1,2,3], **Célestin C. Kokonendji**[1], **Asep Saefuddin**[2]

[1] University of Franche-Comté, Besançon, France.
[2] Bogor Agricultural University, Bogor, Indonesia.
[3] Lampung University, Bandar Lampung, Indonesia.

Abstract. Multivariate normal-Poisson model has been recently introduced as a special case of normal stable Tweedie models. The model is composed of a univariate Poisson variable, and the remaining variables given the Poisson one are independent Gaussian variables with variance the value of the Poisson component. Two characterizations of this model are shown, first by variance function and then by generalized variance function which is the determinant of the variance function. The latter provides an explicit solution of a particular Monge-Ampère equation.

Keywords. Generalized variance, Infinitely divisible measure, Monge-Ampère equation, Multivariate exponential family, Variance function.

MSC: 62H05; 60E07.

1 Introduction

Motivated by normal gamma and normal inverse Gaussian models, Boubacar Maïnassara and Kokonendji (2014) introduced a new form of generalized variance functions which are generated by the so-called *normal stable Tweedie* (NST) models of k-variate

Khoirin Nisa(✉)(khoirin.nisa@univ-fcomte.fr), Célestin C. Kokonendji (celestin.kokonendji@univ-fcomte.fr), Asep Saefuddin (asaefuddin@ipb.ac.id)

distributions ($k > 1$). The generating σ-finite positive measure $\mu_{\alpha,t}$ on $\mathbb{R}^k$ of NST models is composed by the well-known probability measure $\xi_{\alpha,t}$ of univariate positive σ-stable distribution generating L process $\left(X_t^{\alpha}\right)_{t>0}$ which was introduced by Feller (1971) as follows

$$\xi_{\alpha,t}(dx) = \frac{1}{\pi x}\sum_{r=0}^{\infty}\frac{t^r\Gamma(1+\alpha r)sin(-r\pi\alpha)}{r!\alpha^r(\alpha-1)^{-r}\left[(1-\alpha)x\right]^{\alpha r}}\mathbb{1}_{x>0}dx = \xi_{\alpha,t}(x)dx,$$

where $\alpha \in (0,1)$ is the index parameter, $\Gamma(.)$ is the classical gamma function, and $\mathbb{I}_A$ denotes the indicator function of any given event A that takes the value 1 if the event accurs and 0 otherwise. Paremeter α can be extended into $\alpha \in (-\infty, 2]$ (see Tweedie, 1984). For $\alpha = 2$, we obtain the normal distribution with density

$$\xi_{2,t}(dx) = \frac{1}{\sqrt{2\pi t}}\exp\left(\frac{-x^2}{2t}\right)dx.$$

For a k-dimensional NST random vector $\mathbf{X} = (X_1, \dots, X_k)^\top$, the generating σ-finite positive measure $\mu_{\alpha,t}$ is given by

$$\mu_{\alpha,t}(d\mathbf{x}) = \xi_{\alpha,t}(dx_1)\prod_{j=2}^{k}\xi_{2,x_1}(dx_j), \tag{1.1}$$

where X_1 is a univariate (non-negative) stable Tweedie variable and $(X_2, \dots, , X_k)^\top =: \mathbf{X}_1^c$ given X_1 are $k-1$ real independent Gaussian variables with variance X_1.

Normal-Poisson model is a special case of NST models; it is new and the only model which has a discrete component. Among NST models, normal-gamma is the only model which is also a member of simple quadratic natural exponential families (NEFs) of Casalis (1996); she called it "gamma-Gaussian" and it has been characterized by variance and generalized variance functions. See Casalis (1996) or Kotz et al (2000, Chapter 54) for characterization by variance function, and Kokonendji and Masmoudi (2013) for characterization by generalized variance function which is the determinant of covariance matrix expressed in terms of the mean vector.

In contrast to normal-gamma which is the same to gamma-Gaussian; normal-Poisson and Poisson-Gaussian (Kokonendji and Masmoudi , 2006; Koudou and Pommeret , 2002) are two completely different models. Indeed, for any value of $j \in \{1, \dots, k\}$, normal-Poisson model has only one Poisson component and $k-1$ Gaussian components, while a Poisson-Gaussian$_j$ model has j Poisson components and $k-j$ Gaussian components which are all independent. Poisson-Gaussian is a particular case of the simple quadratic NEFs with variance function $\mathbf{V}_F(\boldsymbol{m}) = \mathbf{diag}_k(m_1, \dots, m_j, 1, \dots, 1)$

where $\boldsymbol{m} = (m_1, \ldots, m_k)^\top$ is the mean vector, and its generalized variance function is $\det \mathbf{V}_F(\boldsymbol{m}) = m_1 \ldots m_j$. Some characterizations of Poisson-Gaussian$_j$ models have been done by several authors such as Letac (1989) for variance function, Kokonendji and Masmoudi (2006) for generalized variance function, and Koudou and Pommeret (2002) for convolution-stability. Also one can see Kokonendji and Seshadri (1996); Kokonendji and Pommeret (2007) for the generalized variance estimators of Poisson-Gaussian. This normal-Poisson is also different from the purely discrete "Poisson-normal" model of Steyn (1976), which can be defined as a multiple mixture of k independent Poisson distributions with parameters $m_1, m_2, \ldots, m_k$ and those parameters have a multivariate normal distribution.

Three generalized variance estimators of normal Poisson model have been introduced (Kokonendji and Nisa , 2016). In this paper we present the characterizations of multivariate normal-Poisson model by variance function and by generalized variance function which is connected to the Monge-Ampère equation (Gutiérrez , 2001). In Section 2 we present some properties of normal-Poisson model. We present the characterizations of normal-Poisson model by variance function in Section 3 and the characterization by generalized variance in Section 4.

2 Normal-Poisson model

By introducing "power variance" parameter p defined by $(p-1)(1-\alpha) = 1$ and equivalent to $p = p(\alpha) = \dfrac{\alpha - 2}{\alpha - 1}$ or $\alpha = \alpha(p) = \dfrac{p-2}{p-1}$ (see Jorgensen , 1997, Chapter 4, for complete description of the power unit variance function of univariate stable Tweedie distributions), in the case of $\alpha \to -\infty$ or $p = p(-\infty) = 1$, Expression (1.1) will lead to k-variate normal-Poisson model. Replacing $\alpha(p)$ with $p(\alpha)$ the generating measure of normal-Poisson model can be express as follows

$$\mu_t(d\mathbf{x}) = \mu_{1,t}(d\mathbf{x}) = \xi_{1,t}(dx_1) \prod_{j=2}^{k} \xi_{0,x_1}(dx_j). \tag{2.1}$$

Then by (2.1), for a fixed power of convolution $t > 0$, denote $F_t = F(\mu_t)$ the multivariate NEF (Kotz et al , 2000, Chapter 54) of normal-Poisson with $\mu_t := \mu^{*t}$, the NEF of a k-dimensional normal-Poisson random vector $\mathbf{X}$ is generated by

$$\mu_t(d\mathbf{x}) = \frac{t^{x_1}(x_1!)^{-1}}{(2\pi x_1)^{(k-1)/2}} \exp\left(-t - \frac{1}{2x_1}\sum_{j=2}^{k} x_j^2\right) \mathbb{1}_{x_1 \in \mathbb{N}\setminus\{0\}} \delta_{x_1}(dx_1) \prod_{j=2}^{k} dx_j. \tag{2.2}$$

Since $t > 0$ then μ_t is known to be an infinitely divisible measure; see, e.g., Sato (1999).

The cumulant function which is the logarithm of the Laplace transform of μ_t, i.e. $\mathbf{K}_{\mu_t}(\boldsymbol{\theta}) = \log \int_{\mathbb{R}^k} \exp(\boldsymbol{\theta}^\top \mathbf{x})\mu_t(d\mathbf{x})$, is given by

$$\mathbf{K}_{\mu_t}(\boldsymbol{\theta}) = t \exp\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k} \theta_j^2\right). \tag{2.3}$$

The function $\mathbf{K}_{\mu_t}(\boldsymbol{\theta})$ is finite for all $\boldsymbol{\theta}$ in the canonical domain

$$\boldsymbol{\Theta}(\mu_t) = \left\{\boldsymbol{\theta} \in \mathbb{R}^k; \boldsymbol{\theta}^\top \tilde{\boldsymbol{\theta}}_1^c := \theta_1 + \frac{1}{2}\sum_{j=2}^{k} \theta_j^2 < 0\right\} \tag{2.4}$$

with

$$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^\top \text{ and } \tilde{\boldsymbol{\theta}}_1^c := (1, \theta_2, \ldots, \theta_k)^\top. \tag{2.5}$$

The probability distribution of normal-Poisson which is a member of NEF is given by

$$P(\boldsymbol{\theta}; \mu_t)(d\mathbf{x}) = \exp\{\boldsymbol{\theta}^\top \mathbf{x} - \mathbf{K}_{\mu_t}(\boldsymbol{\theta})\}\mu_t(d\mathbf{x}).$$

From (2.3) we can calculate the first derivative of the cumulant function that produces a k-vector as the mean vector of F_t, and also its second derivative which is a $k \times k$ matrix that represents the covariance matrix. Using notations in (2.5) we obtain

$$\mathbf{K}'_{\mu_t}(\boldsymbol{\theta}) = \mathbf{K}_{\mu_t}(\boldsymbol{\theta}) \times \tilde{\boldsymbol{\theta}}_1^c \quad \text{and} \quad \mathbf{K}''_{\mu_t}(\boldsymbol{\theta}) = \mathbf{K}_{\mu_t}(\boldsymbol{\theta}) \times \left[\tilde{\boldsymbol{\theta}}_1^c \tilde{\boldsymbol{\theta}}_1^{c\top} + \mathbf{I}_k^{0_1}\right],$$

with $\mathbf{I}_k^{0_1} = \mathbf{diag}_k(0, 1, \ldots, 1)$. The cumulant function presented in (2.3) and its derivatives are functions of the canonical parameter $\boldsymbol{\theta}$. For practical calculation we need to use the following mean parameterization:

$$P(\mathbf{m}; F_t) := P(\boldsymbol{\theta}(\mathbf{m}); \mu_t),$$

where $\boldsymbol{\theta}(\mathbf{m})$ is the solution in $\boldsymbol{\theta}$ of the equation $\mathbf{m} = \mathbf{K}'_{\mu_t}(\boldsymbol{\theta})$.

The variance function of normal-Poisson model which is the variance-covariance matrix in term of mean parameterization is obtained through the second derivative of the cumulant function, i.e. $\mathbf{V}_{F_t}(\mathbf{m}) = \mathbf{K}''_{\mu_t}(\boldsymbol{\theta}(\mathbf{m}))$. Then we have

$$\mathbf{V}_{F_t}(\mathbf{m}) = \frac{1}{m_1}\mathbf{m}\mathbf{m}^\top + \mathbf{diag}_k(0, m_1, \ldots, m_1) \tag{2.6}$$

on its support

$$\mathbf{M}_{F_t} = \left\{\mathbf{m} \in \mathbb{R}^k; m_1 > 0 \text{ and } m_j \in \mathbb{R} \text{ for } j = 2, \ldots, k\right\}. \tag{2.7}$$

Consequently, its generalized variance function is

$$\det \mathbf{V}_{F_t}(\mathbf{m}) = m_1^k \text{ with } \mathbf{m} \in \mathbf{M}_{F_t}. \tag{2.8}$$

Equation (2.8) expresses that the generalized variance of normal-Poisson model depends mainly on the mean of the Poisson component.

The infinite divisibility property of normal-Poisson is very useful for its characterization by generalized variance. Regarding to this property, Hassairi (1999) found an interesting representation as stated in the following proposition (without proof).

Proposition 2.1. *If μ is an infinitely divisible measure generating an NEF $F = F(\mu)$ on $\mathbb{R}^k$, then there exists a positive measure $\rho(\mu)$ on $\mathbb{R}^k$ such that*

$$\det \mathbf{K}''_{\mu}(\boldsymbol{\theta}) = \exp \mathbf{K}_{\rho(\mu)}(\boldsymbol{\theta}),$$

for all $\boldsymbol{\theta} \in \Theta(\mu)$. The positive measure $\rho(\mu)$ is called the modified Lévy measure of μ.

For F_t of normal-Poisson model, the modified Lévy measure that satisfies Proposition 2.1 is given by

$$\rho(\mu_t) = t^k \left(\delta_{\mathbf{e}_1} \prod_{j=2}^{k} \mathcal{N}(0,1)\right)^{*k}, \tag{2.9}$$

where $(\mathbf{e}_1)$ an orthonormal basis of $\mathbb{R}^k$ and $\mathcal{N}(0,1)$ is the standard univariate normal probability measure. It comes from the cumulant function of $\rho(\mu_t)$ which is

$$\mathbf{K}_{\rho(\mu_t)}(\boldsymbol{\theta}) = k^t \left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k} \theta_j^2\right)^t =: \left(k \times \boldsymbol{\theta}^\top \tilde{\boldsymbol{\theta}}_1^c\right)^t.$$

By implementing Proposition 2.1 into normal-Poisson model we obtain

$$\det \mathbf{K}''_{\mu_t}(\boldsymbol{\theta}) = t \exp\left(k \times \boldsymbol{\theta}^\top \tilde{\boldsymbol{\theta}}_1^c\right). \tag{2.10}$$

We use (2.10) for characterizing normal-Poisson by generalized variance. The problem in this characterization is that for given information in the right-hand side of (2.10), we need to find the cumulant function $\mathbf{K}$ in the left-hand side of (2.10) such that the determinant of its second derivative equals to the Laplace transform $\exp \mathbf{K}_{\rho(\mu_t)}(\boldsymbol{\theta})$. So, this problem becomes a particular case of the Monge-Ampère equation (see equation (4.1) in Section 4).

3 Characterization by variance function

In order to characterize normal-Poisson model through its generalized variance function from (2.10) back to (2.3) and then to (2.2), it is also interesting to have their characterizations by variance functions from (2.6) back to (2.3), up to some elementary operations of NEFs.

We here state the first result as follows.

Theorem 3.1. *Let $k \in \{2, 3, \ldots\}$ and $t > 0$. If an NEF F_t satisfies (2.6), then, up to affinity, F_t is normal-Poisson model.*

The proof is established by analytical calculations and using the well-known properties of NEFs described in Proposition 3.1 below.

Proposition 3.1. *Let μ and $\widetilde{\mu}$ be two σ-finite positive measures on $\mathbb{R}^k$ such that $F = F(\mu)$, $\widetilde{F} = F(\widetilde{\mu})$ and $\mathbf{m} \in \mathbf{M}_F$.*

(i) If there exists $(\mathbf{d}, c) \in \mathbb{R}^k \times \mathbb{R}$ such that $\widetilde{\mu}(d\mathbf{x}) = \exp\{\langle \mathbf{d}, \mathbf{x}\rangle + c\}\mu(d\mathbf{x})$, then $F = \widetilde{F}$: $\mathbf{\Theta}_{\widetilde{\mu}} = \mathbf{\Theta}_{\mu} - \mathbf{d}$ and $\mathbf{K}_{\widetilde{\mu}}(\boldsymbol{\theta}) = \mathbf{K}_{\mu}(\boldsymbol{\theta} + \mathbf{d}) + c$; for $\widetilde{\mathbf{m}} = \mathbf{m} \in \mathbf{M}_F$,

$$\mathbf{V}_{\widetilde{F}}(\widetilde{\mathbf{m}}) = \mathbf{V}_F(\mathbf{m}) \quad \textit{and} \quad \det \mathbf{V}_{\widetilde{F}}(\widetilde{\mathbf{m}}) = \det \mathbf{V}_F(\mathbf{m}).$$

(ii) If $\widetilde{\mu} = \varphi_* \mu$ with $\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, then: $\mathbf{\Theta}(\widetilde{\mu}) = \mathbf{A}^\top \mathbf{\Theta}(\mu)$ and $\mathbf{K}_{\widetilde{\mu}}(\boldsymbol{\theta}) = \mathbf{K}_{\mu}(\mathbf{A}^\top \boldsymbol{\theta}) + \mathbf{b}^\top \boldsymbol{\theta}$; for $\widetilde{\mathbf{m}} = \mathbf{A}\mathbf{m} + \mathbf{b} \in \varphi(\mathbf{M}_F)$,

$$\mathbf{V}_{\widetilde{F}}(\widetilde{\mathbf{m}}) = \mathbf{A}\mathbf{V}_F(\varphi^{-1}(\widetilde{\mathbf{m}}))\mathbf{A}^\top \quad \textit{and} \quad \det \mathbf{V}_{\widetilde{F}}(\widetilde{\mathbf{m}}) = (\det \mathbf{A})^2 \det \mathbf{V}_{\mathbf{F}}(\mathbf{m}).$$

(iii) If $\widetilde{\mu} = \mu^{*t}$ is the t-th convolution power of μ for $t > 0$, then, for $\widetilde{\mathbf{m}} = t\mathbf{m} \in t\mathbf{M}_F$,

$$\mathbf{V}_{\widetilde{F}}(\widetilde{\mathbf{m}}) = t\mathbf{V}_F(t^{-1}\widetilde{\mathbf{m}}) \quad \textit{and} \quad \det \mathbf{V}_{\widetilde{F}}(\widetilde{\mathbf{m}}) = t^k \det \mathbf{V}_F(\mathbf{m}).$$

Proposition 3.1 shows that the generalized variance function of F, $\det \mathbf{V}_F(\mathbf{m})$, is invariant for any element of its generating measure (Part (i)) and for affine transformation $\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ such that $\det \mathbf{A} = \pm 1$, in particular for a translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{b}$ (Part (ii)). Sometimes we use terminology *type* to call an NEF F as a particular model up to affinity (Part (ii)) and convolution power (Part (iii)).

Proof. Without loss of generality, first we assume that $t = 1$ with flashback to the identifiability of Poisson component.

Let $F = F(\mu)$ be an NEF satisfies (2.6) and (2.7) for $t = 1$. Using the blockwise inversion into $\mathbf{V}_F(\mathbf{m})$ in (2.6), one has:

$$[\mathbf{V}_F(\mathbf{m})]^{-1} = \begin{pmatrix} \frac{1}{m_1} + \frac{1}{m_1^3}\sum_{j=2}^k m_j^2 & -\frac{1}{m_1^2}(\mathbf{m}_1^c)^\top \\ -\frac{1}{m_1^2}(\mathbf{m}_1^c) & \frac{1}{m_1}\mathbf{I}_{k-1} \end{pmatrix} \tag{3.1}$$

with $m_1 > 0$ and $\mathbf{m}_1^c := (m_2, \ldots, m_k)^\top \in \mathbb{R}^k$. Since $\mathbf{m} = \mathbf{K}'_\mu(\boldsymbol{\theta})$ and $\mathbf{V}_F(\mathbf{m}) = \mathbf{K}''_\mu(\boldsymbol{\theta})$, then by writing $\boldsymbol{\theta}$ in terms of $\mathbf{m}$ one gets

$$\mathbf{V}_F(\mathbf{m}) = [\boldsymbol{\theta}'(\mathbf{m})]^{-1}$$

which implies

$$\boldsymbol{\theta}(\mathbf{m}) = \int [\mathbf{V}_F(\mathbf{m})]^{-1}\, d\mathbf{m}.$$

For $\boldsymbol{\theta} \in \boldsymbol{\Theta} := \boldsymbol{\theta}(\mathbf{M}_F)$ such that $\mathbf{M}_F$ has the same elements as $\mathbf{M}_{F_t}$ in (2.7), there exists a function $\varphi : \mathbb{R}^k \to \mathbb{R}$ such that

$$\boldsymbol{\theta}'(\mathbf{m}) = \left[\frac{\partial^2 \varphi(\mathbf{m})}{\partial m_i \partial m_j}\right]_{i,j=1,2,\ldots,k}. \tag{3.2}$$

Using (3.2) into (3.1) for getting the first information on Poisson component, we have

$$\frac{\partial^2 \varphi(\mathbf{m})}{\partial m_1^2} = \frac{1}{m_1} + \frac{1}{m_1^3}\sum_{j=2}^k m_j^2$$

and then

$$\frac{\partial \varphi(\mathbf{m})}{\partial m_1} = \log m_1 - \frac{1}{2m_1^2}\sum_{j=2}^k m_j^2 + f(m_2, \ldots, m_k), \tag{3.3}$$

where $f : \mathbb{R}^{k-1} \to \mathbb{R}$ is an analytical function to be determined. Note that since $m_1 > 0$ then $\log m_1$ and $1/(2m_1^2)$ in (3.3) are well-defined. Derivative of (3.3) with respect to m_j gives

$$\frac{\partial^2 \varphi}{\partial m_1 m_j} = -\frac{m_j}{m_1^2} + \frac{\partial f(m_2, \ldots, m_k)}{\partial m_j}. \tag{3.4}$$

Expression (3.4) is equal to the $(1,j)$th element of $[\mathbf{V}_F(\mathbf{m})]^{-1}$ in (3.1), that is

$$-\frac{m_j}{m_1^2}+\frac{\partial f(m_2,\ldots,m_k)}{\partial m_j}=-\frac{m_j}{m_1^2};$$

therefore, $\partial f(m_2,\ldots,m_k)/\partial m_j=0$ for all $j\in\{2,\ldots,k\}$, this implies $f(m_2,\ldots,m_k)=c_1$ (a real constant). Thus, (3.3) becomes

$$\frac{\partial\varphi}{\partial m_1}=\log m_1-\frac{1}{2m_1^2}\sum_{j=2}^{k}m_j^2+c_1 \tag{3.5}$$

and by integration with respect to m_1, one gets

$$\varphi(\mathbf{m})=m_1\log m_1-m_1+\frac{1}{2m_1}\sum_{j=2}^{k}m_j^2+c_1m_1+h(m_2,\ldots,m_k), \tag{3.6}$$

where $h:\mathbb{R}^{k-1}\to\mathbb{R}$ is an analytical function to be determined. From now on, complete information of the model (i.e. normal components) begin to show itself. The first and second derivatives of (3.6) with respect to m_j give, respectively,

$$\frac{\partial\varphi(\mathbf{m})}{\partial m_j}=\frac{m_j}{m_1}+\frac{\partial h(m_2,\ldots,m_k)}{\partial m_j},\qquad \forall j\in\{2,\ldots,k\} \tag{3.7}$$

and

$$\frac{\partial^2\varphi(\mathbf{m})}{\partial m_j^2}=\frac{1}{m_1}+\frac{\partial h^2(m_2,\ldots,m_k)}{\partial m_j^2},\qquad \forall j\in\{2,\ldots,k\}. \tag{3.8}$$

Expression (3.8) is equal to the diagonal (j,j)th element of $[\mathbf{V}_F(\mathbf{m})]^{-1}$ in (3.1) for all $j\in\{2,\ldots,k\}$, hence we have

$$\frac{1}{m_1}+\frac{\partial^2 h(m_2,\ldots,m_k)}{\partial m_j^2}=\frac{1}{m_1}.$$

Consequently, $\partial^2 h(m_2,\ldots,m_k)/\partial m_j^2=0$ and $\partial h(m_2,\ldots,m_k)/\partial m_j=c_j$ (a real constant) for all $j\in\{2,\ldots,k\}$. Then, equation (3.7) becomes

$$\frac{\partial\varphi(\mathbf{m})}{\partial m_j}=\frac{m_j}{m_1}+c_j\qquad \forall j\in\{2,\ldots,k\}. \tag{3.9}$$

Using equation (3.5) and (3.9) one obtains

$$\boldsymbol{\theta}(\mathbf{m}) = \left(\log m_1 - \frac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2, \quad \frac{m_2}{m_1}, \quad \dots \quad , \quad \frac{m_k}{m_1} \right)^{\top} + (c_1, \dots, c_k)^{\top}$$

or

$$\boldsymbol{\theta}(\mathbf{m}) = \begin{cases} \theta_1 &= \log m_1 - \dfrac{1}{2m_1^2}\sum_{j=2}^{k} m_j^2 + c_1 \\ \theta_j &= \dfrac{m_j}{m_1} + c_j \qquad , \quad j = 2, \dots, k. \end{cases} \tag{3.10}$$

From (3.10), each θ_j belongs to $\mathbb{R}$ for $j \in \{1, 2, \dots, k\}$ because $m_1 > 0$ and $m_j \in \mathbb{R}$ for $j \in \{2, \dots, k\}$. Thus, one has $\boldsymbol{\Theta}(\mathbf{M}_F) =: \boldsymbol{\Theta} \subseteq \mathbb{R}^k$ and also

$$m_1 = \exp\left\{(\theta_1 - c_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - c_j)^2\right\}, \tag{3.11}$$

$$m_j = (\theta_j - c_j)\exp\left\{(\theta_1 - c_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - c_j)^2\right\}. \tag{3.12}$$

Since $\mathbf{m} = \dfrac{\partial \mathbf{K}_\mu(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$, then using (3.11) one can obtain $\mathbf{K}_\mu(\boldsymbol{\theta})$ as follow:

$$\begin{aligned} \mathbf{K}_\mu(\boldsymbol{\theta}) &= \int \frac{\partial \mathbf{K}'_\mu(\boldsymbol{\theta})}{\partial \theta_1} d\theta_1 \\ &= \exp\left\{(\theta_1 - c_1) + \frac{1}{2}\sum_{\ell=2}^{k}(\theta_j - c_j)^2\right\} + g(\theta_2, \dots, \theta_k), \end{aligned} \tag{3.13}$$

where $g : \mathbb{R}^{k-1} \to \mathbb{R}$ is an analytical function to be determined. Again, derivative of (3.13) with respect to θ_j produces

$$\frac{\partial \mathbf{K}_\mu(\boldsymbol{\theta})}{\partial \theta_j} = (\theta_j - c_j)\exp\left\{(\theta_1 - c_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - c_j)^2\right\} + \frac{\partial g(\theta_2, \dots, \theta_k)}{\partial \theta_j}$$

which is equal to (3.12); then, one gets $\partial g(\theta_2, \dots, \theta_k)/\partial \theta_j = 0$ for all $j \in \{2, \dots, k\}$ implying $g(\theta_2, \dots, \theta_k) = C$ (a real constant). Finally, it ensues from it that we have

$$\mathbf{K}_\mu(\boldsymbol{\theta}) = \exp\left\{(\theta_1 - c_1) + \frac{1}{2}\sum_{j=2}^{k}(\theta_j - c_j)^2\right\} + C.$$

By Proposition 3.1 one can see that, up to affinity, this $\mathbf{K}_\mu$ is a normal-Poisson cumulant function as given in (2.3) with $t = 1$ on its corresponding support (2.4). Theorem 3.1 is therefore proven by using the analytical property of $\mathbf{K}_\mu$. □

4 Characterization by generalized variance function

Before stating our next result, let us briefly recall that, for an unknown smooth function $\mathbf{K} : \mathbf{\Theta} \subseteq \mathbb{R}^k \to \mathbb{R}$, $k \geqslant 2$, the MongeAmpère equation is defined by

$$\det \mathbf{K}''(\boldsymbol{\theta}) = g(\boldsymbol{\theta}), \tag{4.1}$$

where $\mathbf{K}'' = \left(D^2_{ij}\mathbf{K}\right)_{i,j=1,\dots,k}$ denotes the Hessian matrix of $\mathbf{K}$ and g is a given positive function (see e.g. Gutiérrez (2001)). The class of equation (4.1) given g has been a source of intense investigations which are related to many areas of mathematics. Note also that explicit solutions of (4.1), even if in particular situations of g, remain generally challenging problems. We can refer to Kokonendji and Seshadri (1996); Kokonendji and Masmoudi (2006, 2013) for some details and handled particular cases.

We now state the next result in the following sense.

Theorem 4.1. *Let $F_t = F(\mu_t)$ be an infinitely divisible NEF on $\mathbb{R}^k$ ($k > 1$) such that*

1. *$\mathbf{\Theta}(\mu_t) = \mathbb{R}^k$, and*
2. *$\det \mathbf{K}''_{\mu_t}(\boldsymbol{\theta}) = t \exp\left(k \times \boldsymbol{\theta}^\top \tilde{\boldsymbol{\theta}}_1^c\right)$*

for $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}_1^c$ given as in (2.5). Then F_t is of normal-Poisson type.

To proof of this theorem is to solve the Monge-Ampère equation problem of normal-Poisson model (item 2 of the theorem). For that purpose, we need three propositions which are already used in Kokonendji and Masmoudi (2006) and Kokonendji and Masmoudi (2013) and we provide the propositions below for making the paper as selfcontained as possible.

Proposition 4.1. *If μ is an infinitely divisible measure on $\mathbb{R}^k$, then there exist a symmetric non-negative definite $d \times d$ matrix Σ with rank $r \leqslant k$ and a positive measure ν on $\mathbb{R}^k$ such that*

$$\mathbf{K}''_\mu(\boldsymbol{\theta}) = \boldsymbol{\Sigma} + \int_{\mathbb{R}^k} \mathbf{x}\mathbf{x}^\top \exp(\boldsymbol{\theta}^\top \mathbf{x})\nu(d\mathbf{x}).$$

(See, e.g. Gikhman and Skorokhod , 2004, page 342).

The above expression of $\mathbf{K}''_\mu(\boldsymbol{\theta})$ is an equivalent of the Lévy-Khinchine formula (see e.g. Sato, 1999); thus, $\boldsymbol{\Sigma}$ comes from a Brownian part and the rest $\mathbf{L}''_\nu(\boldsymbol{\theta}) := \int_{\mathbb{R}^k} \mathbf{x}\mathbf{x}^\top \exp(\boldsymbol{\theta}^\top \mathbf{x})\nu(d\mathbf{x})$ corresponds to jumps part of the associated Lévy process through the Lévy measure ν.

Proposition 4.2. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two* $k \times k$ *matrices. Then*

$$\det(\mathbf{A}+\mathbf{B}) = \sum_{S\subset\{1,2,\ldots,k\}} \det\left(\mathbf{A}_{S'}\right)\det\left(\mathbf{B}_S\right),$$

with $S' = \{1,2,\ldots,k\} \setminus S$ *and* $\mathbf{A}_S = (a_{ij})_{i,j\in S^2}$ *for* $\mathbf{A} = (a_{ij})_{i,j\in\{1,2,\ldots,k\}^2}$. *(See Muir, 1960).*

Proposition 4.3. *Let* $f : \mathbb{R}^k \to \mathbb{R}$ *be a* C^2 *map. Then,* f *is an affine polynomial if and only if*

$$\partial^2 f(\theta)/\partial\theta_i^2 = 0, \quad \text{for} \quad i = 1,\ldots,k.$$

(See Bar-Lev, et al. , 1994, Lemma 4.1).

Proof. Without loss of generality, we assume $t = 1$ in Theorem 4.1. Letting $F = F(\mu)$, we have to solve the following equation (with respect to μ_t or its characteristic function):

$$\det \mathbf{K}''_\mu(\boldsymbol{\theta}) = \exp\left\{k\cdot\left(\theta_1 + \frac{1}{2}\sum_{j=2}^{k}\theta_j^2\right)\right\}, \qquad \forall\boldsymbol{\theta}\in\mathbb{R}^k. \tag{4.2}$$

From Proposition 4.1 relative to the representation of infinitely divisible distribution, the unknown left member of Equation (4.2) can be written as

$$\det \mathbf{K}''_\mu(\boldsymbol{\theta}) = \det\left[\boldsymbol{\Sigma} + \int_{\mathbb{R}^k} \mathbf{x}\mathbf{x}^\top \exp(\boldsymbol{\theta}^\top\mathbf{x})\nu(d\mathbf{x})\right] = \det\left[\boldsymbol{\Sigma} + \mathbf{L}''_\nu(\boldsymbol{\theta})\right]. \tag{4.3}$$

For $S = \{i_1, i_2, \ldots, i_j\}$, with $1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k$, a non-empty subset of $\{1,2,\ldots,k\}$, and $\tau_S : \mathbb{R}^k \to \mathbb{R}^j$ the map defined by $\tau_S(\mathbf{x}) = (x_{i_1}, x_{i_2}, \ldots, x_{i_j})^\top$, we define ν_S the image measure of

$$H_j(d\mathbf{x}_1,\ldots,d\mathbf{x}_j) = \frac{1}{j!}\left(\det\left[\tau_S(\mathbf{x}_1)\ldots\tau_S(\mathbf{x}_j)\right]\right)^2 \nu(d\mathbf{x}_1)\ldots\nu(d\mathbf{x}_j)$$

by $\psi_j : (\mathbb{R}^k)^j \to \mathbb{R}^k, (\mathbf{x}_1,\ldots,\mathbf{x}_j) \mapsto \mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_j$. By Proposition 4.2 and Expression (4.3) the modified Lévy measure $\rho(\mu)$ in (2.1) can be expressed as

$$\rho(\mu) = (\det\boldsymbol{\Lambda})\delta_0 + \sum_{\emptyset\neq S\subset\{1,2,\ldots,k\}} (\det\boldsymbol{\Lambda}_{S'})\nu_S\,, \tag{4.4}$$

where $\mathbf{\Lambda}$ is a diagonal representation of $\mathbf{\Sigma}$ in an orthonormal basis $\mathbf{e} = (\mathbf{e}_i)_{i=1,\dots,k}$ (see Hassairi , 1999, page 384). Since $\mathbf{\Sigma}$ is the Brownian part, then it corresponds to the $k-1$ normal components from the right member of (4.2); that implies $r = \mathrm{rank}(\mathbf{\Sigma}) = k-1$ and $\det \mathbf{\Sigma} = 0$. Therefore $\det \mathbf{\Lambda} = 0$ with $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$ such that $\lambda_1 = 0$ and $\lambda_j > 0$ for all $j \in \{2, \dots, k\}$. For all non-empty subsets S of $\{1, 2, \dots, k\}$ there exist real numbers $\alpha_S \geqslant 0$ such that

$$(\det \mathbf{\Lambda}_{S'})\nu_S = \left(\prod_{i \notin S} \lambda_i\right) \nu_S = \alpha_S \left[\delta_{\mathbf{e}_1} * \mathcal{N}(0,1)(\mathbf{e}_1^c)\right]^{*k}, \tag{4.5}$$

where $\mathbf{e}_1^c = (\mathbf{e}_2, \dots, \mathbf{e}_k)$ denotes the induced orthonormal basis of $\mathbf{e}$ without component $\mathbf{e}_1$; i.e. $k-1$ is the dimension of $\mathbf{e}_1^c$.

With respect to Kokonendji and Masmoudi (2006, Lemma 7) for making precise the measure ν of (4.5), it is easy to see that $S_0 = \{1\}$ is a singleton (i.e. set with exactly one element) such that, for $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_k\mathbf{e}_k$,

$$x_1^2 \nu(d\mathbf{x}) = \beta \delta_{a\mathbf{e}_1},$$

with $\beta > 0$ and $a \neq 0$. Consequently, we have the following complementary set $S_0' = \{1, 2, \dots, k\} \setminus \{1\}$. So, from (4.5) we have kth power of convolution of only one Poisson at the first component $\mathbf{e}_1$ and $(k-1)$-variate standard normal. That means $\mathbf{K}_\mu''(\boldsymbol{\theta}) = \mathbf{K}_\mu(\boldsymbol{\theta})\left[\tilde{\boldsymbol{\theta}}_1^c \tilde{\boldsymbol{\theta}}_1^{c\top} + \mathbf{I}_k^{0_1}\right]$, with notations of (2.5). Let $\mathbf{B}(\boldsymbol{\theta}) = \exp\left(\theta_1 + \frac{1}{2}\sum_{j=2}^k \theta_j^2\right)$ from (4.2). Since we check that $\partial^2(\mathbf{K}_\mu - \mathbf{B})(\boldsymbol{\theta})/\partial\theta_i^2 = 0$ for all $i = 1, \dots, k$, Proposition 4.3 allows that $(\mathbf{K}_{\mu_1} - \mathbf{B})(\boldsymbol{\theta})$ is an affine function on $\mathbb{R}^k$ and therefore

$$\mathbf{K}_\mu(\boldsymbol{\theta}) = \exp\left(\theta_1 + \frac{1}{2}\sum_{j=2}^k \theta_j^2\right) + \mathbf{u}^\top \boldsymbol{\theta} + b,$$

for $(\mathbf{u}, b) \in \mathbb{R}^k \times \mathbb{R}$. Hence $F = F(\mu)$ is of normal-Poisson type with $t = 1$. This completes the proof of the theorem. □

A reformulation of Theorem 4.1, by changing the canonical parameterization into the mean parameterization, is stated in the following theorem without proof.

Theorem 4.2. *Let $F_t = F(\mu_t)$ be an infinitely divisible NEF on $\mathbb{R}^k$ such that*

1. $\mathbf{M}_{F_t} = \left\{\mathbf{m} \in \mathbb{R}^k; m_1 > 0 \text{ and } m_j \in \mathbb{R} \text{ with } j = 2, \dots, k\right\}$, *and*

2. $\det \mathbf{V}_{F_t}(\mathbf{m}) = m_1^k$.

Then F_t is of normal-Poisson type.

Theorem 4.1 can be viewed as the solution to a particular Monge-Ampère equation (4.1). Whereas Theorem 4.2 is interesting for generalized variance estimation of the model.

5 Conclusion

In this paper we described some properties of normal-Poisson model. Then we showed that the characterization of normal-Poisson model by variance function was obtained through analytical calculations and using some properties of NEF. Also, the characterization of normal Poisson model by generalized variance which is the solution to a specific Monge-Ampère equation: $\det \mathbf{K}''_\mu(\boldsymbol{\theta}) = \exp\left(k \times \boldsymbol{\theta}^\top \tilde{\boldsymbol{\theta}}_1^c\right)$ on $\mathbb{R}^k$ can be solved using the infinite divisibility property of normal-Poisson.

Acknowledgment

The authors would like to thank the Editor and Referees for their valuable and constructive comments that led to a considerably improved manuscript.

References

Bar-Lev, S., Bschouty, D., Enis, P., Letac, G., Lu, I. and Richard, D. (1994), The diagonal multivariate natural exponential families and their classification. *Journal of Theoretical Probability*, **7** (4), 883-929.

Boubacar Maïnassara, Y. and Kokonendji, C. C. (2014), Normal stable Tweedie models and power-generalized variance function of only one component, *TEST*, **23** (3) , 585-606.

Casalis, M. (1996), The $2d + 4$ simple quadratic natural exponential families on $\mathbb{R}^d$. *The Annals of Statistics*, **24** (4), 1828-1854.

Feller, W. (1971), *An Introduction to Probability Theory and its Applications* Vol. II Second edition. New York: Wiley.

Gikhman, I. I. and Skorokhod, A.V. (2004), *The Theory of Stochastic Processes II*. New York: Springer.

Gutiérrez, C.E. (2001), *The Monge-Ampère Equation*. Boston: Birkhäuser.

Hassairi, A. (1999), Generalized variance and exponential families. *The Annals of Statistics*, **27** (1), 374-385.

Jorgensen, B. (1997). *The Theory of Dispersion Models.* London: Chapman & Hall.

Kokonendji, C.C. and Masmoudi, A. (2006), A characterization of Poisson-Gaussian families by generalized variance. *Bernoulli*, **12** (2), 371-379.

Kokonendji, C.C. and Masmoudi, A. (2013), On the Monge-Ampère equation for characterizing gamma-Gaussian model. *Statistics and Probability Letters*, **83** (7), 1692-1698.

Kokonendji, C. C. and Nisa, K. (2016). Generalized variance estimations of normal Poisson models. In *Forging Connections between Computational Mathematics and Computational Geometry, Springer Proceeding in Mathematics and Statistics* **124**, Eds. K. Chen and A. Ravindran. Switzerland: Springer, pp. 247-260.

Kokonendji, C.C. and Pommeret, D. (2007), Comparing UMVU and ML estimators of the generalized variance for natural exponential families. *Statistics*, **41** (6), 547-558.

Kokonendji, C.C. and Seshadri, V. (1996), On the determinant of the second derivative of a Laplace transform. *The Annals of Statistics*, **24** (4), 1813-1827.

Kotz, S., Balakrishnan, N. and Johnson, N.L. (2000), *Continuous Multivariate Distributions Vol.1: Models and Applications. 2nd edition*. New York: Wiley.

Koudou, A.E. and Pommeret, D. (2002), A characterization of Poisson-Gaussian families by convolution-stability. *Journal of Multivariate Analysis*, **81**, 120-127.

Letac, G. (1989), Le problème de la classification des familles exponentielles naturelles sur $\mathbb{R}^d$ ayant une fonction variance quadratique. *In Probability Measures on Groups IX - Lecture Notes in Mathematics*. **1306**, Ed. H. Heyer. Berlin: Springer 194-215.

Muir, T. (1960), *A Treatise on the Theory of Determinants*. New York: Dover.

Sato, K. (1999), *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

Steyn, H.S. (1976), On the multivariate Poisson normal distribution. *Journal of the American Statistical Association*, **71** (353), 233-236.

Tweedie, M.C.K. (1984), An index which distinguishes between some important exponential families. In: *Statistics: Applications New Directions, Proceedings of the Indian Statistical Golden Jubilee International Conference*, Eds. J. K. Ghosh, J. Roy. Calcutta: Indian Statistical Institute, 579-604.