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On Concomitants of Order Statistics and its Application in Defining Ranked Set Sampling from Farlie-Gumbel-Morgenstern Bivariate Lomax Distribution

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Abstract. In this paper, we have dealt with the distribution theory of concomitants of order statistics arising from Farlie-Gumbel-Morgenstern bivariate Lomax distribution. We have discussed the estimation of the parameters associated with the distribution of the variable Y of primary interest, based on the ranked set sample defined by ordering the marginal observations on an auxiliary variable X, when (X, Y) follows a Farlie-Gumbel-Morgenstern bivariate Lomax distribution. When the association parameter and the shape parameter corresponding to Y are known, we have proposed four estimators, viz., an unbiased estimator based on the Stokes' ranked set sample, the best linear unbiased estimator based on the Stokes' ranked set sample, the best linear unbiased estimator based on the extreme ranked set sample and the best linear unbiased estimator based on the multistage extreme ranked set sample for the scale parameter of the variable of primary interest. The relative efficiencies of these estimators have also been worked out.

Keywords. Best linear unbiased estimator, Extreme ranked set sampling, Fisher information, Multistage ranked set sampling

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1 Introduction

In modelling bivariate data, sometimes, prior information will be available about the form of marginals of the parent bivariate distribution. Johnson and Kotz (1972) have introduced a class of bivariate distributions known as Farlie-Gumbel-Morgenstern (FGM) distributions in which each cumulative distribution function (cdf) $F(x, y)$ is determined by the specified marginal cdfs $F_X(x)$ and $F_Y(y)$ and possesses a form given by

$$F(x, y) = F_X(x)F_Y(y)\{1 + \alpha[1 - F_X(x)][1 - F_Y(y)]\}, \; -1 \leq \alpha \leq 1, \tag{1.1}$$

where α is known as the association parameter. The probability density function (pdf) corresponding to the bivariate cdf $F(x, y)$ defined in (1.1) is given by

$$f(x, y) = f_X(x)f_Y(y)\{1 + \alpha[1 - 2F_X(x)][1 - 2F_Y(y)]\}, \; -1 \leq \alpha \leq 1, \tag{1.2}$$

where $f_X(.)$ and $f_Y(.)$ are the pdfs corresponding to the marginal cdfs $F_X(.)$ and $F_Y(.)$, respectively. It is clear to note that, if we put $\alpha = 0$ in (1.1), then it describes the case of the independence of distributions of marginal random variables X and Y of (X, Y) with cdf $F(x, y)$.

The concept of concomitants of order statistics was first introduced by David (1973). Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be a random sample drawn from an arbitrary bivariate distribution with cdf $F(x, y)$ and pdf $f(x, y)$. If the sample values on the marginal random variable X are ordered as $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, then the accompanying Y-observation in the ordered pair with X-observation equal to the rth order statistic $X_{r:n}$ is called the concomitant of $X_{r:n}$ and is denoted by $Y_{[r:n]}$. For a review of results on concomitants of order statistics and their applications, see David and Nagaraja (1998). For some recent developments in the theory of concomitants of order statistics, see Thomas and Veena (2011) and Veena and Thomas (2008, 2015, 2016).

One of the important applications of concomitants of order statistics is in the ranked set sampling (RSS). McIntyre (1952) introduced RSS in situations where observational economy consideration is warranted in the experimentation involved. For a review of various types of RSS and their applications in parameter estimation, see Chen *et al.* (2004). Ranking of units by judgement method as required in McIntyre's method of RSS becomes unsuitable when one is not confident of making perfect ranking on the units or when ambiguity arises in discriminating the rank of one unit with another. In such situations, one may prefer ranking of units based on the measurements made on some easily measurable variable (auxiliary variable) that is correlated with the variable of primary interest (study variable). Thus, as an alternative to McIntyre's method of RSS, Stokes (1977) has defined a RSS procedure by fixing the units based on the rank assigned to the observations made on an auxiliary variable X which is jointly distributed with the study variable Y and used the resulting sample to estimate the mean of the variable Y. Chen *et al.* (2004) used Stokes' RSS to estimate the total amount of plutonium present in the surface soil within a fenced area adjacent to a hazard waste site. In Stokes' method of RSS, n independent sets of units each of size n are selected. In the first set, the Y variable of the unit associated with the smallest ordered X is measured. In the second set, the Y variable of the unit associated with the second smallest X is measured. This procedure is continued until the Y variable of the unit associated with the largest X from the nth set is measured. Thus, the observations made on the n units selected by the procedure explained above constitute a ranked set sample of size n.

Stokes (1977) suggested the ranked set sample mean as an estimator for the mean of the study variate Y when an auxiliary variate X is used for ranking the sampling units under the assumption that (X, Y) follows a bivariate normal distribution. Barnett and Moore (1997) improved it by deriving the best linear combination of the observations of the ranked set sample as an unbiased estimator of the mean of Y.

Since the FGM family of bivariate distributions is a flexible family possessing a wide variety of models, it gives ample freedom for a user to choose the most appropriate one for modelling any bivariate data set. Also, as it is easier to construct a bivariate distribution with prior information available about the form of the marginal distributions, extensive studies have been undertaken on this family both in the theoretical framework and in the application perspective. Chacko and Thomas (2008, 2009) have obtained the best linear unbiased estimators (BLUEs) of the parameters involved in the distribution of Y using ranked set sample, when (X, Y) follows FGM bivariate exponential distribution and when (X, Y) follows FGM bivariate logistic distribution. For some other recent works in this direction, see Chacko and Thomas (2007), Lesitha *et al.* (2010) and Lesitha and Thomas (2013).

Al-Saleh and Al-Kadiri (2000) have introduced the concept of double stage ranked set sampling (DSRSS) as an extension of the RSS introduced by McIntyre (1952) and shown that the estimator based on the DSRSS is more efficient than those estimators based on both RSS and simple random sampling in estimating the population parameters. Al-Saleh and Al-Omari (2002) have generalized DSRSS to multistage ranked set sampling (MSRSS) and proved that there is increase in the precision of MSRSS estimators when compared with DSRSS estimators and RSS estimators. Chacko and Thomas (2008) have obtained the BLUE of the mean of the study variate Y using multistage ranked set sample, when (X, Y) follows FGM bivariate exponential distribution.

Even though estimation of parameters of several distributions belonging to the FGM family have been carried out using RSS, the inference problems of Farlie-Gumbel-Morgenstern bivariate Lomax (FGMBL) distribution using RSS is not seen carried out so far in the available literature. Hence the aim of this paper is to consider the distributional aspects of concomitants of order statistics arising from FGMBL distribution and to deal with the problem of estimation of some parameters involved in this distribution.

For details on the genesis of Lomax distribution and its applications for the study of business failure data, see Lomax (1954). Turnbull *et al.* (1974) have used Lomax distribution for the analysis of heart transplant data and Fox and Kraemer (1971) have carried out the study of remission rates of psychiatric patients using this distribution.

The joint pdf $h(x, y)$ of the FGMBL distribution is obtained by substituting the pdfs $f_X(x) = \frac{\beta_1\lambda_1^{\beta_1}}{(\lambda_1+x)^{\beta_1+1}}$, $f_Y(y) = \frac{\beta_2\lambda_2^{\beta_2}}{(\lambda_2+y)^{\beta_2+1}}$ and cdfs $F_X(x) = 1 - (\frac{\lambda_1}{\lambda_1+x})^{\beta_1}$, $F_Y(y) = 1 - (\frac{\lambda_2}{\lambda_2+y})^{\beta_2}$ of two univariate Lomax distributions in (1.2). Thus we have

$$h(x, y) = \frac{\beta_1\beta_2\lambda_1^{\beta_1}\lambda_2^{\beta_2}}{(\lambda_1 + x)^{\beta_1+1}(\lambda_2 + y)^{\beta_2+1}} \left\{1 + \alpha\, [2(\frac{\lambda_1}{\lambda_1 + x})^{\beta_1} - 1]\, [2(\frac{\lambda_2}{\lambda_2 + y})^{\beta_2} - 1]\right\}, \tag{1.3}$$

where $x, y > 0$; $\lambda_1, \lambda_2 > 0$; $\beta_1, \beta_2 > 0; -1 \leq \alpha \leq 1$ and $h(x, y)$ takes zero elsewhere. Here α is the association parameter, λ_1 and λ_2 are the scale parameters and β_1 and β_2 are the shape parameters.

Clearly,

$$E(X) = \frac{\lambda_1}{\beta_1 - 1}, \quad V(X) = \frac{\beta_1\lambda_1^2}{(\beta_1 - 1)^2(\beta_1 - 2)},$$

$$E(Y) = \frac{\lambda_2}{\beta_2 - 1}, \tag{1.4}$$

$$V(Y) = \frac{\beta_2\lambda_2^2}{(\beta_2 - 1)^2(\beta_2 - 2)} \tag{1.5}$$

and the correlation coefficient ρ between X and Y is given by

$$\rho = \alpha\frac{\sqrt{\beta_1\beta_2(\beta_1 - 2)(\beta_2 - 2)}}{(2\beta_1 - 1)(2\beta_2 - 1)}. \tag{1.6}$$

The main objective of this paper is to develop the distribution theory of concomitants of order statistics arising from FGMBL distribution so as to expose this model to users to deal further with the theoretical as well as applied perspectives to problems involving concomitants of order statistics.

This paper is organized as follows. The distribution theory of concomitants of order statistics arising from the FGMBL distribution is developed in Section 2. It may be noted that the distribution theory of concomitants of order statistics is an essential requirement for applying Stokes' RSS. Thus, we use the results developed in Section 2 to generate ranked set sample from FGMBL distribution and the details are given in Section 3. In this section, four types of estimators of the scale parameter λ_2 involved in the FGMBL distribution have been proposed when the association parameter α and the shape parameter β_2 are known. Two of them are (i) an unbiased estimator $\widetilde{\lambda}_2$ based on the mean of the observations of the Stokes' ranked set sample and (ii) the BLUE $\widehat{\lambda}_2$ based on the Stokes' ranked set sample. Fisher information (FI) contained in the concomitant of a particular order statistic of a random sample drawn from a distribution gives the required knowledge for selecting the most appropriate unit(s) from a group of units to define an appropriate ranked set sample. Thus the FI about the parameter λ_2 contained in the concomitant of rth order statistic of a random sample of size n arising from the FGMBL distribution has been derived and presented in Section 3.3. The values of FI contained in all concomitants in a sample of size n are computed for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = 0.25(0.25)1$ and are presented in Table 1. Using the computed values in Table 1, we have identified the concomitant which contains the maximum amount of information as the concomitant of the smallest order statistic or the largest order statistic according as $\alpha < 0$ or $\alpha > 0$. Accordingly, we define an extreme ranked set sampling (ERSS) in Section 3.3 and used the observations in it to propose another estimator λ_2^* of the scale parameter λ_2 when α and β_2 are known. We have also proposed an unbiased estimate $\widehat{\widehat{\lambda}}_2$ of λ_2 based on the method of moments approach using a simple random sample of equivalent sample size n arising from the FGMBL distribution. The efficiencies of λ_2^* relative to $\widetilde{\lambda}_2$, $\widehat{\lambda}_2$ relative to $\widetilde{\lambda}_2$ and λ_2^* relative to $\widehat{\widehat{\lambda}}_2$ are computed for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = 0.25(0.25)1$ and are presented in Table 2.

In some real life situations such as in the study of bilirubin level of newborn babies with liver complaints, one may measure the bilirubin level (auxiliary variable X) in the urine of all babies without pain and much effort whereas making measurement on the bilirubin level of blood (variable Y of primary interest) is painful and sometimes risky. In particular, in such situations measurement on the auxiliary variable X can be made from any number of units whereas number of units to be chosen for making measurement on the variable Y of primary interest should be as small as possible. Multistage ranked set sampling is a very suitable sampling strategy in the above described cases and accordingly we have utilized Section 3.4 to define multistage extreme ranked set sampling (MSERSS) and propose an estimator $\widehat{\lambda}_2^{n(r)}$ of λ_2 which is the BLUE based on the observations in the sample generated by MSERSS. We have further analysed the steady state efficiency of $\widehat{\lambda}_2^{n(r)}$ (as $r \to \infty$). Finally, an estimator of $1/(\beta_2 - 1)$ has been proposed in Section 4 using Stokes' RSS when λ_2 is known. Keeping in mind the method of moments approach of estimating parameters, we have thereby proposed an estimate of β_2 as well in this section.

2 Distribution theory of concomitants of order statistics arising from FGMBL distribution

Let (X_i, Y_i), $i = 1, 2, \ldots, n$, be a random sample of size n drawn from the FGMBL distribution defined by the pdf (1.3). Let $Y_{[r:n]}$ be the concomitant of the rth order statistic $X_{r:n}$. Then using the expression for the distribution of concomitants of order statistics given by Scaria and Nair (1999), the pdf $h_{[r:n]}(y)$ of $Y_{[r:n]}$, $1 \le r \le n$, and the joint pdf $h_{[r,s:n]}(y_1, y_2)$ of $Y_{[r:n]}$ and $Y_{[s:n]}$, $1 \le r < s \le n$, are obtained as

$$h_{[r:n]}(y) = \frac{\beta_2 \lambda_2^{\beta_2}}{(\lambda_2 + y)^{\beta_2+1}} \left\{1 + \frac{\alpha(n-2r+1)}{n+1}[2(\frac{\lambda_2}{\lambda_2+y})^{\beta_2} - 1]\right\}, \tag{2.1}$$

$$y > 0;\ \lambda_2 > 0;\ \beta_2 > 0;\ -1 \le \alpha \le 1$$

and

$$\begin{aligned} &h_{[r,s:n]}(y_1, y_2) \\ &= \frac{\beta_2^2 \lambda_2^{2\beta_2}}{(\lambda_2 + y_1)^{\beta_2+1}(\lambda_2 + y_2)^{\beta_2+1}} \left\{1 + \frac{\alpha(n-2r+1)}{n+1}[2(\frac{\lambda_2}{\lambda_2+y_1})^{\beta_2} - 1]\right. \\ &+ \frac{\alpha(n-2s+1)}{n+1}[2(\frac{\lambda_2}{\lambda_2+y_2})^{\beta_2} - 1] \\ &\left. + \alpha^2\{\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\}[2(\frac{\lambda_2}{\lambda_2+y_1})^{\beta_2} - 1][2(\frac{\lambda_2}{\lambda_2+y_2})^{\beta_2} - 1]\right\}, \end{aligned}$$

$$y_1, y_2 > 0;\ \lambda_2 > 0;\ \beta_2 > 0;\ -1 \le \alpha \le 1.$$

The expressions for the means, variances and covariances of the concomitants of order statistics $Y_{[r:n]}$, $1 \le r \le n$, arising from the FGMBL distribution with pdf defined by (1.3) are obtained as follows.

For $1 \le r \le n$,

$$E(Y_{[r:n]}) = \lambda_2 \left\{ \frac{1}{\beta_2 - 1} + \frac{\alpha\beta_2(n-2r+1)}{n+1} \left[\frac{2\,\mathbf{\Gamma}(2\beta_2 - 1)}{\mathbf{\Gamma}(2\beta_2 + 1)} - \frac{\mathbf{\Gamma}(\beta_2 - 1)}{\mathbf{\Gamma}(\beta_2 + 1)} \right] \right\}, \tag{2.2}$$

provided $\beta_2 > 1$, where $\mathbf{\Gamma}(.)$ is the complete gamma function. For $1 \le r \le n$,

$$\begin{aligned} Var(Y_{[r:n]}) =& \lambda_2^2 \left\{ \frac{2}{(\beta_2-1)(\beta_2-2)} + \frac{2\alpha\beta_2(n-2r+1)}{n+1} \left(\frac{2\,\mathbf{\Gamma}(2\beta_2-2)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-2)}{\mathbf{\Gamma}(\beta_2+1)} \right) \right. \\ & \left. - \left[\frac{1}{\beta_2-1} + \frac{\alpha\beta_2(n-2r+1)}{n+1} \left(\frac{2\,\mathbf{\Gamma}(2\beta_2-1)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-1)}{\mathbf{\Gamma}(\beta_2+1)} \right) \right]^2 \right\}, \end{aligned} \tag{2.3}$$

provided $\beta_2 > 2$. For $1 \le r < s \le n$,

$$\begin{aligned} Cov(Y_{[r:n]}, Y_{[s:n]}) = & \lambda_2^2 \alpha^2 \beta_2^2 \left[\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)} - \frac{(n-2r+1)(n-2s+1)}{(n+1)^2} \right] \\ & \times \left[\frac{2\,\mathbf{\Gamma}(2\beta_2-1)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-1)}{\mathbf{\Gamma}(\beta_2+1)} \right]^2, \text{ provided } \beta_2 > 1. \end{aligned} \tag{2.4}$$

Now, for $\beta_2 > 2$, we define the constants $\psi_{r,n}$, $\delta_{r,r,n}$ and $\delta_{r,s,n}$ by

$$\psi_{r,n} = \frac{1}{\beta_2-1} + \frac{\alpha\beta_2(n-2r+1)}{n+1} \left[\frac{2\,\mathbf{\Gamma}(2\beta_2-1)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-1)}{\mathbf{\Gamma}(\beta_2+1)} \right], \tag{2.5}$$

$$\begin{aligned} \delta_{r,r,n} =& \frac{2}{(\beta_2-1)(\beta_2-2)} + \frac{2\alpha\beta_2(n-2r+1)}{n+1} \left(\frac{2\,\mathbf{\Gamma}(2\beta_2-2)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-2)}{\mathbf{\Gamma}(\beta_2+1)} \right) \\ & - \left[\frac{1}{\beta_2-1} + \frac{\alpha\beta_2(n-2r+1)}{n+1} \left(\frac{2\,\mathbf{\Gamma}(2\beta_2-1)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-1)}{\mathbf{\Gamma}(\beta_2+1)} \right) \right]^2 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \delta_{r,s,n} = & \alpha^2 \beta_2^2 \left[\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)} - \frac{(n-2r+1)(n-2s+1)}{(n+1)^2} \right] \\ & \times \left[\frac{2\,\mathbf{\Gamma}(2\beta_2-1)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-1)}{\mathbf{\Gamma}(\beta_2+1)} \right]^2. \end{aligned} \tag{2.7}$$

Consequently, we may re-write equations (2.2), (2.3) and (2.4) as, for $1 \leq r \leq n$,

$$E(Y_{[r:n]}) = \lambda_2 \, \psi_{r,n} \,, \tag{2.8}$$

$$Var(Y_{[r:n]}) = \lambda_2^2 \, \delta_{r,r,n} \tag{2.9}$$

and for $1 \leq r < s \leq n$,

$$Cov(Y_{[r:n]}, Y_{[s:n]}) = \lambda_2^2 \, \delta_{r,s,n}.$$

Clearly, the constants $\psi_{r,n}$, $\delta_{r,r,n}$ and $\delta_{r,s,n}$ are known whenever α and β_2 are known.

3 Estimation of λ_2 when α and β_2 are known

In this section we consider the situation where the association parameter α and the shape parameter β_2 are known and derive four types of estimators of λ_2 of the FGMBL distribution viz. (i) an unbiased estimator based on the observations in the Stokes' ranked set sample, (ii) the BLUE based on the observations in the Stokes' ranked set sample, (iii) the BLUE based on the observations in the extreme ranked set sample and (iv) the BLUE based on the observations in an unbalanced multistage ranked set sample.

3.1 Unbiased estimator of λ_2 using Stokes' RSS

Let (X, Y) be a bivariate random variable which has an FGMBL distribution with pdf defined by (1.3) where α and β_2 are assumed to be known. Suppose n sets of sampling units, each of size n are drawn from the FGMBL distribution. We assume that making measurement on the variable X is cheap and easy. Hence from each of the units in the rth set we make measurement on the variable X, order them as $X_{(1:n)r}, X_{(2:n)r}, \ldots, X_{(n:n)r}$ and now choose the unit with X-observation equal to $X_{(r:n)r}$ to make measurement on Y. Let the value of Y measured on this unit be denoted by $Y_{[r:n]r}$. If we adopt this process with all n sets, then it generates the ranked set sample $Y_{[1:n]1}, Y_{[2:n]2}, \ldots, Y_{[n:n]n}$. Clearly, the distribution of $Y_{[r:n]r}$ is the same as that of $Y_{[r:n]}$, the concomitant of rth order statistic of a random sample of size n arising from the distribution (1.3). By using the respective expressions (2.8) and (2.9) for the means and variances of concomitants of order statistics arising from the FGMBL distribution, we obtain the means and variances of $Y_{[r:n]r}$, for $1 \leq r \leq n$, as

$$E(Y_{[r:n]r}) = \lambda_2 \, \psi_{r,n} \tag{3.1}$$

and

$$Var(Y_{[r:n]r}) = \lambda_2^2\,\delta_{r,r,n}, \tag{3.2}$$

where $\psi_{r,n}$ and $\delta_{r,r,n}$ are given as in (2.5) and (2.6) respectively. Since $Y_{[r:n]r}$ and $Y_{[s:n]s}$ (for $r \neq s$) are measurements on Y made from units involved in two independent samples, we have

$$Cov(Y_{[r:n]r}, Y_{[s:n]s}) = 0, \text{ for } r \neq s. \tag{3.3}$$

The following theorem gives an unbiased estimator $\widetilde{\lambda}_2$ of λ_2 using ranked set sample observations obtained by Stokes' method.

Theorem 3.1. *Let (X, Y) have FGMBL distribution with pdf given by (1.3). Let $Y_{[r:n]r}$, $r = 1, 2, \ldots, n$, be the ranked set sample observations on the variable Y of primary interest. Then*

$$\widetilde{\lambda}_2 = \frac{(\beta_2 - 1)}{n}\sum_{r=1}^{n} Y_{[r:n]r} \tag{3.4}$$

is an unbiased estimator of λ_2 and its variance is given by

$$Var(\widetilde{\lambda}_2) = \frac{\lambda_2^2(\beta_2-1)^2}{n}\left\{\frac{2}{(\beta_2-1)(\beta_2-2)} - \frac{1}{(\beta_2-1)^2} - \frac{\alpha^2{\beta_2}^2(n-1)}{3(n+1)}\left[\frac{2\,\mathbf{\Gamma}(2\beta_2-1)}{\mathbf{\Gamma}(2\beta_2+1)} - \frac{\mathbf{\Gamma}(\beta_2-1)}{\mathbf{\Gamma}(\beta_2+1)}\right]^2\right\}, \tag{3.5}$$

provided $\beta_2 > 2$.

Proof. The proof of the theorem follows as a straightforward implication of equations (3.1) to (3.4). □

Remark 1. If variance of $\widetilde{\lambda}_2$ for a given value of $\alpha \in (0, 1]$ is evaluated, then this variance is equal to the variance of $\widetilde{\lambda}_2$ for $-\alpha$ since the expression (3.5) for the variance of $\widetilde{\lambda}_2$ depends on α by a term involving α^2 only.

Remark 2. When α is unknown, using (1.6) we propose a moment type estimator for α as follows. For FGMBL distribution, the correlation coefficient between the variables X and Y is given by $\rho = \alpha\frac{\sqrt{\beta_1\beta_2(\beta_1-2)(\beta_2-2)}}{(2\beta_1-1)(2\beta_2-1)} = \alpha K$ where $K = \frac{\sqrt{\beta_1\beta_2(\beta_1-2)(\beta_2-2)}}{(2\beta_1-1)(2\beta_2-1)}$, where $\beta_1 \neq \frac{1}{2}$ and $\beta_2 \neq \frac{1}{2}$. If we use the pooled set of observations made on X from all units available, then the unknown quantity β_1 may be estimated using the first and third quartiles of those X observations. Let P_1 and P_3 denote the first and third quartiles of the distribution of the marginal random variable X. Then

$$\frac{1}{4} = P(X \leq P_1) = 1 - (\frac{\lambda_1}{\lambda_1 + P_1})^{\beta_1}$$

and

$$\frac{3}{4} = P(X \leq P_3) = 1 - (\frac{\lambda_1}{\lambda_1 + P_3})^{\beta_1}.$$

On simplification we obtain

$$\frac{P_1}{(4/3)^{(1/\beta_1)} - 1} = \frac{P_3}{4^{(1/\beta_1)} - 1}. \tag{3.6}$$

If we replace P_1 and P_3 in (3.6) by $\hat{P_1}$ and $\hat{P_3}$, the first and third sample quartiles respectively for the data on X-values, then on solving for β_1, we get an estimate $\beta_1^{(0)}$ of β_1.

If n is moderately large and we consider $Y_{[r:n]r}$, $r = 1, 2, ..., n$, as a sample of size n from the distribution of the marginal random variable Y and write Q_1 and Q_3 to denote the first and third quartiles of the marginal distribution of Y, we then get an equation

$$\frac{Q_1}{(4/3)^{(1/\beta_2)} - 1} = \frac{Q_3}{4^{(1/\beta_2)} - 1}. \tag{3.7}$$

In (3.7), if we replace Q_1 and Q_3 by $\hat{Q_1}$ and $\hat{Q_3}$, the first and third quartiles of the sample $Y_{[1:n]1}, Y_{[2:n]2}, \ldots, Y_{[n:n]n}$, then we can solve for β_2. Let $\beta_2^{(0)}$ be the estimate obtained.

Then using $\beta_1^{(0)}$ and $\beta_2^{(0)}$, we can estimate K as $K^{(0)} = \frac{\sqrt{\beta_1^{(0)}\beta_2^{(0)}(\beta_1^{(0)}-2)(\beta_2^{(0)}-2)}}{(2\beta_1^{(0)}-1)(2\beta_2^{(0)}-1)}$. If r is the sample correlation coefficient between $X_{(i:n)i}$ and $Y_{[i:n]i}$, $i = 1, 2, ..., n$, then the moment type estimator of α is obtained by equating r with the population correlation coefficient ρ and is obtained as

$$\hat{\alpha} = \begin{cases} -1, \text{ if } r \leq -K^{(0)} \\ \frac{r}{K^{(0)}}, \text{ if } -K^{(0)} < r < K^{(0)} \\ 1, \quad \text{if } r \geq K^{(0)}. \end{cases}$$

Thus, when all λ_2, β_2 and α are unknown, we can estimate those parameters step by step as described above.

3.2 BLUE of λ_2 using Stokes' RSS

In this section, we provide a better estimator $\widehat{\lambda_2}$ of λ_2 by deriving the BLUE when α and β_2 are known.

Suppose n sets of sampling units each of size n are drawn from the FGMBL distribution with pdf defined by (1.3) and generated a ranked set sample as proposed by Stokes (1977). Let $\mathbf{Y}_{[n]} = (Y_{[1:n]1}, Y_{[2:n]2}, \ldots, Y_{[n:n]n})'$ be the column vector of concomitants of order statistics arising from (1.3). Clearly, the distribution of $Y_{[r:n]r}$ is the same as that of $Y_{[r:n]}$. Then we may write from equation (2.8), the mean vector of $\mathbf{Y}_{[n]}$ as

$$E(\mathbf{Y}_{[n]}) = \lambda_2 \, \mathbf{\Psi}, \tag{3.8}$$

where $\mathbf{\Psi} = (\psi_{1,n}, \psi_{2,n}, \ldots, \psi_{n,n})'$ and $\psi_{r,n}$ is as given in (2.5) for $r = 1, 2, \ldots, n$. From equations (2.9) and (3.3), the variance-covariance matrix of $\mathbf{Y}_{[n]}$ may be written as

$$D(\mathbf{Y}_{[n]}) = \lambda_2^2 \mathbf{\Delta}, \tag{3.9}$$

where $\mathbf{\Delta} = diag(\delta_{1,1,n}, \delta_{2,2,n}, \cdots, \delta_{n,n,n})$ and $\delta_{r,r,n}$ is given by (2.6) for $r = 1, 2, \ldots, n$. If α and β_2 are known, then (3.8) and (3.9) together defines a generalized Gauss-Markov set up and the BLUE $\widehat{\lambda_2}$ of λ_2 based on ranked set sample observations is obtained as (see, David and Nagaraja (2003), p.185),

$$\widehat{\lambda_2} = (\mathbf{\Psi}' \mathbf{\Delta}^{-1} \mathbf{\Psi})^{-1} \mathbf{\Psi}' \mathbf{\Delta}^{-1} \mathbf{Y}_{[n]}$$

with variance given by

$$Var(\widehat{\lambda_2}) = \frac{1}{\mathbf{\Psi}' \mathbf{\Delta}^{-1} \mathbf{\Psi}} \lambda_2^2.$$

On simplification we obtain

$$\widehat{\lambda_2} = \frac{\sum_{r=1}^{n} \frac{\psi_{r,n}}{\delta_{r,r,n}} Y_{[r:n]r}}{\sum_{r=1}^{n} \frac{\psi_{r,n}^2}{\delta_{r,r,n}}}$$

with variance given by

$$Var(\widehat{\lambda_2}) = \left\{ \sum_{r=1}^{n} \frac{\psi_{r,n}^2}{\delta_{r,r,n}} \right\}^{-1} \lambda_2^2. \tag{3.10}$$

In this case also, Remark 2 applies as such if one intends in the estimation of all parameters λ_2, β_2 and α in a stage by stage manner.

Remark 3. As in the case of variance of $\widetilde{\lambda}_2$, once the variance of $\widehat{\lambda}_2$ for a given value of $\alpha \in (0,1]$ is evaluated, there is no need to again evaluate the variance for $-\alpha$. To clarify this, we state and prove the following theorem.

Theorem 3.2. *If, for $\alpha \in (0,1]$, $Var^{(\alpha)}(\widehat{\lambda}_2)$ is the variance of the BLUE $\widehat{\lambda}_2$ of λ_2 of the FGMBL distribution with pdf defined by (1.3), then*

$$Var^{(-\alpha)}(\widehat{\lambda}_2) = Var^{(\alpha)}(\widehat{\lambda}_2).$$

Proof. The constants $\psi_{r,n}$ and $\delta_{r,r,n}$ given by (2.5) and (2.6) are functions of α, r and n and hence $\psi_{r,n}$ and $\delta_{r,r,n}$ can be written as $\psi_{r,n}(\alpha)$ and $\delta_{r,r,n}(\alpha)$ respectively. From (2.5) and (2.6), it is obvious that, for $1 \leq r \leq n$,

$$\psi_{r,n}(\alpha) = \psi_{n-r+1,n}(-\alpha) \tag{3.11}$$

and

$$\delta_{r,r,n}(\alpha) = \delta_{n-r+1,n-r+1,n}(-\alpha). \tag{3.12}$$

As a consequence of (3.11) and (3.12), we can rewrite (3.10) as

$$\begin{aligned} Var^{(\alpha)}(\widehat{\lambda}_2) &= \left\{ \sum_{r=1}^{n} \frac{\psi_{r,n}^2(\alpha)}{\delta_{r,r,n}(\alpha)} \right\}^{-1} \lambda_2^2 \\ &= \left\{ \sum_{r=1}^{n} \frac{\psi_{n-r+1,n}^2(-\alpha)}{\delta_{n-r+1,n-r+1,n}(-\alpha)} \right\}^{-1} \lambda_2^2 \\ &= Var^{(-\alpha)}(\widehat{\lambda}_2). \end{aligned}$$

□

3.3 BLUE of λ_2 using ERSS

In order to identify the most suitable unit from a group of units for making measurements on them, we derive the FI about the scale parameter λ_2 contained in the concomitant of rth order statistic arising from FGMBL distribution.

Let $Y_{[r:n]}$, $r = 1, 2, \ldots, n$, be the concomitants of order statistics of a random sample of size n arising from the FGMBL distribution with pdf defined by (1.3). Then, taking the natural logarithm on both sides of the pdf $h_{[r:n]}(y)$ defined in (2.1) we obtain

$$\begin{aligned}\ln h_{[r:n]}(y) &= \ln\beta_2 + \beta_2 \ln\lambda_2 - (\beta_2+1)\ln(\lambda_2+y)\\ &\quad + \ln\left\{1+\frac{\alpha(n-2r+1)}{n+1}[2(\frac{\lambda_2}{\lambda_2+y})^{\beta_2}-1]\right\}. \end{aligned} \tag{3.13}$$

Let the Fisher's measure of information about the parameter λ_2 contained in $Y_{[r:n]}$, the concomitant of rth order statistic $X_{r:n}$ arising from FGMBL distribution with association parameter α be denoted by $I^{(\alpha)}_{\lambda_2}(Y_{[r:n]})$. Then,

$$\begin{aligned} I^{(\alpha)}_{\lambda_2}(Y_{[r:n]}) &= E\Big(\frac{\partial \ln h_{[r:n]}(y)}{\partial\lambda_2}\Big)^2\\ &= \int_0^\infty \Big(\frac{\partial \ln h_{[r:n]}(y)}{\partial\lambda_2}\Big)^2 h_{[r:n]}(y)dy. \end{aligned}$$

Using (2.1) and (3.13) in the above integral and simplifying we get

$$\begin{aligned} I^{(\alpha)}_{\lambda_2}(Y_{[r:n]}) &= \frac{1}{\lambda_2^2}\int_0^\infty \left\{\frac{\beta_2^3}{(1+t)^{\beta_2+1}} + \frac{\beta_2(\beta_2+1)^2}{(1+t)^{\beta_2+3}} - \frac{2\beta_2^2(\beta_2+1)}{(1+t)^{\beta_2+2}}\right\} \times \left\{1+d[\frac{2}{(1+t)^{\beta_2}}-1]\right\} dt\\ &\quad + \frac{1}{\lambda_2^2}\int_0^\infty \left\{\frac{4d\beta_2^2(\beta_2 t-1)t}{(1+t)^{2\beta_2+3}} + \frac{4d^2\beta_2^3 t^2}{(1+t)^{3\beta_2+3}[1+d[\frac{2}{(1+t)^{\beta_2}}-1]]}\right\} dt, \end{aligned} \tag{3.14}$$

where $d = \frac{\alpha(n-2r+1)}{n+1}$.

Theorem 3.3. *If $I^{(\alpha)}_{\lambda_2}(Y_{[r:n]})$ is the FI about λ_2 contained in $Y_{[r:n]}$, the concomitant of rth order statistic $X_{r:n}$ arising from FGMBL distribution with association parameter α, then*

$$I^{(-\alpha)}_{\lambda_2}(Y_{[n-r+1:n]}) = I^{(\alpha)}_{\lambda_2}(Y_{[r:n]}),\ r = 1, 2, \ldots, n. \tag{3.15}$$

Proof. The constant d involved in $I^{(\alpha)}_{\lambda_2}(Y_{[r:n]})$ given by (3.14) is a function of α, r and n and hence one can write d as $d(\alpha, r, n) = \frac{\alpha(n-2r+1)}{n+1}$. It can be easily seen that $d(\alpha, r, n) = d(-\alpha, n-r+1, n)$. Hence the proof of the theorem follows from the expression for $I^{(\alpha)}_{\lambda_2}(Y_{[r:n]})$ given by (3.14). □

We have computed $\lambda_2^2 I_{\lambda_2}^{(\alpha)}(Y_{[r:n]})$ for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = 0.25(0.25)1$ and the computed values are given in Table 1. As a consequence of Theorem 3.3, $I_{\lambda_2}^{(-\alpha)}(Y_{[n-r+1:n]})$ is the same as $I_{\lambda_2}^{(\alpha)}(Y_{[r:n]})$ for the corresponding positive values of α and hence values of $\lambda_2^2 I_{\lambda_2}^{(-\alpha)}(Y_{[n-r+1:n]})$ for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = 0.25(0.25)1$ are not included in the table.

From Table 1, it can be seen that, for $n \leq 10$, the maximum FI about λ_2 is available on the concomitant of largest order statistic when $\alpha > 0$. As a consequence of Theorem 3.3, it follows that the maximum FI about λ_2 is available on the concomitant of smallest order statistic when $\alpha < 0$.

Accordingly, we define a RSS procedure which utilizes either the concomitant of smallest or largest order statistic according as $\alpha < 0$ or $\alpha > 0$ and the problem of estimation of λ_2 based on ERSS which arises in two cases of α is described below.

Case 1: $\alpha < 0$. Let $X_{(1:n)r}$ denote the smallest order statistic of X-observations in the rth sample and $Y_{[1:n]r}$ the measurement made on the Y-variate of the same unit for $r = 1, 2, \ldots, n$. In this case, from Theorem 3.3 and Table 1, we observe that the maximum information about λ_2 is contained in the concomitant of the smallest order statistic and hence we consider a lower extreme ranked set sampling (LERSS) which yields $Y_{[1:n]1}, Y_{[1:n]2}, \cdots, Y_{[1:n]n}$ as the observations in the sample.

Let $\mathbf{Y}_{[n]}^* = (Y_{[1:n]1}, Y_{[1:n]2}, \ldots, Y_{[1:n]n})'$ be the column vector of lower extreme ranked set sample observations. Then we may write the mean vector of $\mathbf{Y}_{[n]}^*$ as

$$E(\mathbf{Y}_{[n]}^*) = \lambda_2 \, \mathbf{\Psi}^*, \tag{3.16}$$

where $\mathbf{\Psi}^* = (\psi_{1,n}, \psi_{1,n}, \cdots, \psi_{1,n})'$ and the variance-covariance matrix of $\mathbf{Y}_{[n]}^*$ may be written as

$$D(\mathbf{Y^*}_{[n]}) = \lambda_2^2 \mathbf{\Delta}^*, \tag{3.17}$$

where $\mathbf{\Delta}^* = diag(\delta_{1,1,n}, \delta_{1,1,n}, \cdots, \delta_{1,1,n})$. Now (3.16) and (3.17) together defines a generalized Gauss-Markov set up and hence the BLUE of λ_2 is given by

$$\lambda_2^* = (\mathbf{\Psi}^{*\prime} \mathbf{\Delta}^{*-1} \mathbf{\Psi}^*)^{-1} \mathbf{\Psi}^{*\prime} \mathbf{\Delta}^{*-1} \mathbf{Y}_{[n]}^*$$

with variance given by

$$Var(\lambda_2^*) = \frac{1}{\mathbf{\Psi}^{*\prime} \mathbf{\Delta}^{*-1} \mathbf{\Psi}^*} \lambda_2^2.$$

On simplifying we obtain

$$\lambda_2^* = \frac{1}{n\psi_{1,n}} \sum_{r=1}^{n} Y_{[1:n]r}$$

with variance given by

$$Var(\lambda_2^*) = \frac{\delta_{1,1,n}}{n\psi_{1,n}^2} \lambda_2^2. \tag{3.18}$$

Table 1: Computed values of $\lambda_2^2 I_{\lambda_2}^{(\alpha)}(Y_{[r:n]})$ for $n = 2(2)10, \beta_2 = 3(1)5$ and $\alpha = 0.25(0.25)1$

n	r	β_2	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
2	1	3	0.58724	0.57761	0.57124	0.56833
	2		0.61579	0.63456	0.65630	0.68104
	1	4	0.64979	0.63629	0.62632	0.62010
	2		0.68680	0.71013	0.73662	0.76630
	1	5	0.69436	0.67797	0.66529	0.65656
	2		0.73762	0.76428	0.79424	0.82751
4	1	3	0.57928	0.56906	0.57057	0.58595
	2		0.59198	0.58506	0.57928	0.57466
	3		0.60911	0.61931	0.63057	0.64290
	4		0.63057	0.67078	0.72084	0.78144
	1	4	0.63871	0.62212	0.61831	0.62976
	2		0.65615	0.64682	0.63871	0.63187
	3		0.67836	0.69121	0.70521	0.72034
	4		0.70521	0.75404	0.81332	0.88372
	1	5	0.68096	0.65956	0.65165	0.65993
	2		0.70192	0.69080	0.68096	0.67244
	3		0.72788	0.74269	0.75868	0.77587
	4		0.75868	0.81380	0.87975	0.95720
6	1	3	0.57650	0.56817	0.57746	0.60931
	2		0.58417	0.57356	0.56852	0.56957
	3		0.59416	0.58888	0.58417	0.58004
	4		0.60640	0.61335	0.62085	0.62890
	5		0.62085	0.64662	0.67732	0.71306
	6		0.63749	0.68867	0.75410	0.83560
	1	4	0.63464	0.61906	0.62276	0.65143
	2		0.64558	0.63015	0.62075	0.61799
	3		0.65903	0.65200	0.64558	0.63980
	4		0.67490	0.68373	0.69314	0.70314
	5		0.69314	0.72488	0.76186	0.80419
	6		0.71372	0.77537	0.85211	0.94573
	1	5	0.67593	0.65484	0.65415	0.68007
	2		0.68931	0.67025	0.65756	0.65187
	3		0.70532	0.69699	0.68931	0.68228
	4		0.72387	0.73408	0.74490	0.75633
	5		0.74490	0.78100	0.82255	0.86965
	6		0.76836	0.83763	0.92253	1.02482
8	1	3	0.57512	0.56843	0.58345	0.62785
	2		0.58047	0.56988	0.56913	0.57976
	3		0.58724	0.57761	0.57124	0.56833
	4		0.59541	0.59115	0.58724	0.58368
	5		0.60493	0.61019	0.61579	0.62172
	6		0.61579	0.63456	0.65630	0.68104
	7		0.62798	0.66421	0.70884	0.76232
	8		0.64148	0.69922	0.77415	0.86915
	1	4	0.63256	0.61819	0.62762	0.66972
	2		0.64041	0.62382	0.61795	0.62456

	3		0.64979	0.63629	0.62632	0.62010
	4		0.66068	0.65505	0.64979	0.64491
	5		0.67302	0.67973	0.68680	0.69422
	6		0.68680	0.71013	0.73662	0.76630
	7		0.70200	0.74616	0.79922	0.86163
	8		0.71861	0.78788	0.87531	0.98378
	1	5	0.67332	0.65310	0.65808	0.69798
	2		0.68303	0.66193	0.65213	0.65555
	3		0.69436	0.67797	0.66529	0.65656
	4		0.70726	0.70062	0.69436	0.68850
	5		0.72169	0.72947	0.73762	0.74614
	6		0.73762	0.76428	0.79424	0.82751
	7		0.75503	0.80496	0.86414	0.93298
	8		0.77390	0.85155	0.94799	1.06608
10	1	3	0.57430	0.56890	0.58815	0.64223
	2		0.57835	0.56859	0.57225	0.59215
	3		0.58337	0.57265	0.56824	0.57082
	4		0.58933	0.58074	0.57430	0.57010
	5		0.59622	0.59266	0.58933	0.58624
	6		0.60401	0.60824	0.61270	0.61737
	7		0.61270	0.62739	0.64408	0.66275
	8		0.62227	0.65008	0.68344	0.72251
	9		0.63273	0.67632	0.73104	0.79787
	10		0.64408	0.70618	0.78752	0.89197
	1	4	0.63131	0.61797	0.63169	0.68428
	2		0.63737	0.62094	0.61916	0.63528
	3		0.64449	0.62868	0.61974	0.61842
	4		0.65261	0.64080	0.63131	0.62424
	5		0.66174	0.65705	0.65261	0.64842
	6		0.67184	0.67725	0.68290	0.68879
	7		0.68290	0.70128	0.72178	0.74440
	8		0.69492	0.72908	0.76916	0.81528
	9		0.70788	0.76066	0.82526	0.90261
	10		0.72178	0.79609	0.89072	1.00953
	1	5	0.67173	0.65234	0.66163	0.71252
	2		0.67931	0.65784	0.65184	0.66486
	3		0.68798	0.66837	0.65598	0.65164
	4		0.69773	0.68351	0.67173	0.66250
	5		0.70851	0.70299	0.69773	0.69272
	6		0.72032	0.72660	0.73313	0.73991
	7		0.73313	0.75420	0.77749	0.80298
	8		0.74693	0.78574	0.83070	0.88193
	9		0.76172	0.82121	0.89295	0.97785
	10		0.77749	0.86067	0.96485	1.09391

Case 2: $\alpha > 0$. Let $X_{(n:n)r}$ denote the largest order statistic of X-observations in the rth sample and $Y_{[n:n]r}$ the measurement made on the Y-variate of the same unit for $r = 1, 2, \dots, n$. In this case, from Theorem 3.3 and Table 1, we observe that the maximum information about λ_2 is contained in the concomitant of the largest order statistic and hence we consider an upper extreme ranked set sampling (UERSS) which yields $Y_{[n:n]1}, Y_{[n:n]2}, \cdots, Y_{[n:n]n}$ as the observations in the sample.

Let $\mathbf{Y}^*_{[n]} = (Y_{[n:n]1}, Y_{[n:n]2}, \cdots, Y_{[n:n]n})'$ be the column vector of upper extreme ranked set sample observations. Then, the BLUE of λ_2 is given by

$$\lambda_2^* = \frac{1}{n\psi_{n,n}} \sum_{r=1}^{n} Y_{[n:n]r}$$

with variance given by

$$Var(\lambda_2^*) = \frac{\delta_{n,n,n}}{n\psi_{n,n}^2}\lambda_2^2. \tag{3.19}$$

Remark 4. Using equations (3.11) and (3.12), it follows from (3.18) and (3.19) that $Var(\lambda_2^*)$ for positive values of α and that for corresponding negative values of α are identically equal.

Using expressions (3.5), (3.10) and (3.19), we have computed the efficiencies $e_1 = e(\lambda_2^*/\widetilde{\lambda}_2) = \frac{Var(\widetilde{\lambda}_2)}{Var(\lambda_2^*)}$ of λ_2^* relative to $\widetilde{\lambda}_2$ and $e_2 = e(\widehat{\lambda}_2/\widetilde{\lambda}_2) = \frac{Var(\widetilde{\lambda}_2)}{Var(\widehat{\lambda}_2)}$ of $\widehat{\lambda}_2$ relative to $\widetilde{\lambda}_2$ for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = 0.25(0.25)1$ and are presented in Table 2. Because of Theorem 3.2, Theorem 3.2 and Remark 4, for $n = 2(2)10$ and $\beta_2 = 3(1)5$, e_1 and e_2 for $\alpha = -1, -0.75, -0.50, -0.25$ are the same as e_1 and e_2 for the corresponding positive values of α and for the same values of n and β_2 and hence the values of e_1 and e_2 for negative values of α are not included in this table.

Whenever we define a new estimation technique, it is customary to study the advantage of the new estimator when compared with a traditional estimator based on a simple random sample. In this respect we now consider an unbiased estimator of λ_2 using the method of moments technique. From (1.4), we notice that an unbiased estimate $\widehat{\widehat{\lambda}}_2$ of λ_2 based on a random sample of equivalent sample size n arising from the marginal distribution of Y is

$$\widehat{\widehat{\lambda}}_2 = (\beta_2 - 1)\bar{Y}, \tag{3.20}$$

where $\beta_2 > 1$ and $\bar{Y}$ is the mean of marginal Y observations of the bivariate random sample. Further, using (1.5) and (3.20) we may write the variance of estimator $\widehat{\widehat{\lambda}}_2$ as

$$Var(\widehat{\widehat{\lambda}}_2) = \frac{\beta_2\lambda_2^2}{n(\beta_2 - 2)}, \tag{3.21}$$

provided $\beta_2 > 2$.

Using expressions (3.19) and (3.21), we have computed the efficiency e_3 of the best estimator λ_2^* relative to $\widehat{\widehat{\lambda}}_2$ given by $e_3 = \frac{Var(\widehat{\widehat{\lambda}}_2)}{Var(\lambda_2^*)}$ for $\beta_2 = 3(1)5$, $n = 2(2)10$ and $\alpha = 0.25(0.25)1$ and are further given in Table 2. We observe that our estimate λ_2^* is highly efficient when compared with $\widehat{\widehat{\lambda}}_2$ for all values of n, β_2 and α tried.

Table 2: The efficiencies $e_1 = e(\lambda_2^*/\widetilde{\lambda}_2)$, $e_2 = e(\widehat{\lambda}_2/\widetilde{\lambda}_2)$ and $e_3 = e(\lambda_2^*/\widehat{\widehat{\lambda}}_2)$ for $\beta_2 = 3(1)5$, $n = 2(2)10$ and $\alpha = 0.25(0.25)1$.

β_2	n	Efficiencies	For the values of α 0.25	0.50	0.75	1
3	2	e_1	1.03354	1.06723	1.10070	1.13362
		e_2	1.00028	1.00116	1.00277	1.00533
		e_3	1.03440	1.07080	1.10901	1.14894
	4	e_1	1.06177	1.12597	1.19123	1.25641
		e_2	1.00051	1.00231	1.00632	1.01490
		e_3	1.06337	1.13277	1.20753	1.28731
	6	e_1	1.07423	1.15235	1.23239	1.31274
		e_2	1.00062	1.00289	1.00831	1.02200
		e_3	1.07615	1.16064	1.25252	1.35135
	8	e_1	1.08123	1.16730	1.25583	1.34496
		e_2	1.00067	1.00319	1.00958	1.02729
		e_3	1.08334	1.17645	1.27820	1.38815
	10	e_1	1.08573	1.17693	1.27096	1.36580
		e_2	1.00072	1.00342	1.01042	1.03143
		e_3	1.08795	1.18663	1.29480	1.41201
4	2	e_1	1.03578	1.07142	1.10650	1.14064
		e_2	1.00015	1.00060	1.00145	1.00288
		e_3	1.03696	1.07630	1.11790	1.16171
	4	e_1	1.06606	1.13459	1.20414	1.27346
		e_2	1.00026	1.00127	1.00369	1.00936
		e_3	1.06824	1.14393	1.22668	1.31645
	6	e_1	1.07948	1.16321	1.24918	1.33565
		e_2	1.00031	1.00159	1.00504	1.01433
		e_3	1.08209	1.17461	1.27709	1.38965
	8	e_1	1.08703	1.17950	1.27502	1.37158
		e_2	1.00036	1.00180	1.00588	1.01802
		e_3	1.08992	1.19212	1.30612	1.43220
	10	e_1	1.09189	1.19002	1.29178	1.39497
		e_2	1.00037	1.00194	1.00646	1.02086
		e_3	1.09493	1.20341	1.32496	1.45998
5	2	e_1	1.03705	1.07383	1.10992	1.14490
		e_2	1.00009	1.00037	1.00091	1.00184
		e_3	1.03838	1.07938	1.12291	1.16895
	4	e_1	1.06850	1.13958	1.21184	1.28395
		e_2	1.00017	1.00082	1.00257	1.00694
		e_3	1.07098	1.15024	1.23763	1.33334
	6	e_1	1.08245	1.16953	1.25925	1.34985
		e_2	1.00019	1.00108	1.00363	1.01097
		e_3	1.08545	1.18257	1.29128	1.41212
	8	e_1	1.09033	1.18662	1.28657	1.38816
		e_2	1.00023	1.00123	1.00430	1.01396
		e_3	1.09359	1.20101	1.32225	1.45815
	10	e_1	1.09539	1.19766	1.30432	1.41320
		e_2	1.00024	1.00129	1.00473	1.01624
		e_3	1.09888	1.21300	1.34249	1.48840

From Table 2, for all values of α and β_2, it can be easily seen that λ_2^* is relatively more efficient than $\widetilde{\lambda}_2$ and $\widehat{\lambda}_2$ is relatively more efficient than $\widetilde{\lambda}_2$. Moreover, the efficiency of the estimator λ_2^* is larger than the BLUE $\widehat{\lambda}_2$ based on Stokes' RSS. Also it can be observed that for a fixed sample size n and β_2, e_1 and e_2 increase as α increases for positive values of α whereas e_1 and e_2 increase as α decreases for negative values of α. The same trend can be observed on the efficiency e_3 as well.

3.4 Estimation of λ_2 using unbalanced MSERSS

The MSRSS scheme (in r stages) is described as follows:

1. Randomly select n^{r+1} units from the target bivariate population, where r is the number of stages of MSRSS and allocate them randomly into n^{r-1} sets, each of size n^2 .
2. For each set in step 1, apply Stokes' RSS procedure described in Section 3.1 to obtain a ranked set sample of size n. This step yields n^{r-1} ranked sets of size n each.
3. Arrange the n^{r-1} ranked sets, each of size n obtained from step 2 randomly into n^{r-2} sets, each of size n^2. Without doing any actual quantification, apply Stokes' RSS method on each of the n^{r-2} sets to yield n^{r-2} second stage ranked sets of size n each.
4. This process is continued, without any actual quantification, until we end up with the rth stage ranked set sample of size n.
5. Finally, the n identified units in step 4 are quantified for the variable of interest.

In Section 3.2, we have considered an RSS method for estimating λ_2 using $Y_{[r:n]r}$ measured on the study variate Y on the unit having the rth smallest value observed on the auxiliary variable X of the rth sample, $r = 1, 2, \ldots, n$ and hence the RSS so considered was balanced. In Section 3.3, it has been proved that in a bivariate sample of size n drawn from the FGMBL distribution, the concomitant of largest order statistic possesses the maximum FI about λ_2 when $\alpha > 0$ and the concomitant of smallest order statistic possesses the maximum FI about λ_2 when $\alpha < 0$. By assuming that the random variable (X,Y) has a FGMBL distribution with pdf (1.3), where Y is the variable of primary interest and X is an auxiliary variable, in this section for $\alpha > 0$, first we consider a multistage upper extreme ranked set sampling (MSUERSS) based on the measurements made on an auxiliary variate to choose the ranked sets and estimate λ_2 involved in the FGMBL distribution based on the measurement made on the variable of primary interest. At each stage and from each set, we choose a unit with the largest value on the auxiliary variable as the units of ranked sets with an objective of exploiting the maximum FI about λ_2 on the finally selected ranked set sample.

Let $U_i^{(r)}$, $i = 1, 2, \ldots, n$, be the units chosen by the (r stage) MSUERSS. Since the measurement of auxiliary variable on each unit $U_i^{(r)}$, $i = 1, 2, \ldots, n$, has the largest value, we may write $Y_{[n:n]i}^{(r)}$ to denote the value measured on the variable of primary interest on $U_i^{(r)}$, $i = 1, 2, \ldots, n$. Clearly, the distribution of $Y_{[n:n]i}^{(r)}$ is the same as that of $Y_{[n^r:n^r]}$, the concomitant of the largest order statistic of n^r independently and identically distributed (*iid*) bivariate random variables with FGMBL distribution. Moreover $Y_{[n:n]i}^{(r)}$, $i = 1, 2, \ldots, n$ are also independently distributed with pdf given by (see, Scaria and Nair (1999))

$$h_{[n:n]i}^{(r)}(y) = \frac{\beta_2 \lambda_2^{\beta_2}}{(\lambda_2 + y)^{\beta_2+1}} \left\{1 - \frac{\alpha(n^r - 1)}{n^r + 1}[2(\frac{\lambda_2}{\lambda_2 + y})^{\beta_2} - 1]\right\}, \tag{3.22}$$

$$y > 0;\ \lambda_2 > 0;\ \beta_2 > 0;\ \alpha > 0.$$

The means and variances of $Y_{[n:n]i}^{(r)}$, for $1 \leq i \leq n$, are obtained as

$$E(Y_{[n:n]i}^{(r)}) = \lambda_2 \left\{\frac{1}{\beta_2 - 1} - \frac{\alpha\beta_2(n^r - 1)}{n^r + 1}\left[\frac{2\,\boldsymbol{\Gamma}(2\beta_2 - 1)}{\boldsymbol{\Gamma}(2\beta_2 + 1)} - \frac{\boldsymbol{\Gamma}(\beta_2 - 1)}{\boldsymbol{\Gamma}(\beta_2 + 1)}\right]\right\}, \tag{3.23}$$

provided $\beta_2 > 1$ and

$$\begin{aligned} Var(Y_{[n:n]i}^{(r)}) =& \lambda_2^2 \left\{\frac{2}{(\beta_2 - 1)(\beta_2 - 2)} - \frac{2\alpha\beta_2(n^r - 1)}{n^r + 1}\left(\frac{2\,\boldsymbol{\Gamma}(2\beta_2 - 2)}{\boldsymbol{\Gamma}(2\beta_2 + 1)} - \frac{\boldsymbol{\Gamma}(\beta_2 - 2)}{\boldsymbol{\Gamma}(\beta_2 + 1)}\right)\right. \\ &\left. - \left[\frac{1}{\beta_2 - 1} - \frac{\alpha\beta_2(n^r - 1)}{n^r + 1}\left(\frac{2\,\boldsymbol{\Gamma}(2\beta_2 - 1)}{\boldsymbol{\Gamma}(2\beta_2 + 1)} - \frac{\boldsymbol{\Gamma}(\beta_2 - 1)}{\boldsymbol{\Gamma}(\beta_2 + 1)}\right)\right]^2\right\}, \end{aligned} \tag{3.24}$$

provided $\beta_2 > 2$.

If, for $\beta_2 > 2$, we write

$$\psi_{n^r,n^r} = \frac{1}{\beta_2 - 1} - \frac{\alpha\beta_2(n^r - 1)}{n^r + 1}\left[\frac{2\,\boldsymbol{\Gamma}(2\beta_2 - 1)}{\boldsymbol{\Gamma}(2\beta_2 + 1)} - \frac{\boldsymbol{\Gamma}(\beta_2 - 1)}{\boldsymbol{\Gamma}(\beta_2 + 1)}\right] \tag{3.25}$$

and

$$\delta_{n^r,n^r,n^r} = \frac{2}{(\beta_2-1)(\beta_2-2)} - \frac{2\alpha\beta_2(n^r-1)}{n^r+1}\left(\frac{2\,\Gamma(2\beta_2-2)}{\Gamma(2\beta_2+1)} - \frac{\Gamma(\beta_2-2)}{\Gamma(\beta_2+1)}\right) - \left[\frac{1}{\beta_2-1} - \frac{\alpha\beta_2(n^r-1)}{n^r+1}\left(\frac{2\,\Gamma(2\beta_2-1)}{\Gamma(2\beta_2+1)} - \frac{\Gamma(\beta_2-1)}{\Gamma(\beta_2+1)}\right)\right]^2, \tag{3.26}$$

then, for $1 \le i \le n$, (3.23) and (3.24) may be re-written as

$$E(Y^{(r)}_{[n:n]i}) = \lambda_2 \psi_{n^r,n^r} \tag{3.27}$$

and

$$Var(Y^{(r)}_{[n:n]i}) = \lambda_2^2 \delta_{n^r,n^r,n^r}. \tag{3.28}$$

Since $Y^{(r)}_{[n:n]i}$ and $Y^{(r)}_{[n:n]j}$ (for $i \neq j$) are measurements on Y made from units involved in two independent samples, we have

$$Cov(Y^{(r)}_{[n:n]i}, Y^{(r)}_{[n:n]j}) = 0, \text{ for } i \neq j. \tag{3.29}$$

Let $\mathbf{Y}^{(r)}_{[n]} = (Y^{(r)}_{[n:n]1}, Y^{(r)}_{[n:n]2}, \ldots, Y^{(r)}_{[n:n]n})'$. Then from equation (3.27), we get the mean vector of $\mathbf{Y}^{(r)}_{[n]}$ as

$$E(\mathbf{Y}^{(r)}_{[n]}) = \lambda_2\, \psi_{n^r,n^r} \mathbf{1}, \tag{3.30}$$

and from equations (3.28) and (3.29), the variance-covariance matrix of $\mathbf{Y}^{(r)}_{[n]}$ may be obtained as

$$D(\mathbf{Y}^{(r)}_{[n]}) = \lambda_2^2 \delta_{n^r,n^r,n^r} \mathbf{I}, \tag{3.31}$$

where $\mathbf{1}$ is a column vector of n ones and $\mathbf{I}$ is a unit matrix of order n. If $\alpha > 0$ and $\beta_2 > 2$ involved in ψ_{n^r,n^r} and δ_{n^r,n^r,n^r} are known, then (3.30) and (3.31) together defines a generalized Gauss-Markov set up and hence the BLUE of λ_2 is obtained as

$$\widehat{\lambda}_2^{n(r)} = \frac{1}{n\psi_{n^r,n^r}} \sum_{i=1}^{n} Y^{(r)}_{[n:n]i} \tag{3.32}$$

with variance given by

$$Var(\widehat{\lambda}_2^{n(r)}) = \frac{\delta_{n^r,n^r,n^r}}{n\psi^2_{n^r,n^r}} \lambda_2^2. \tag{3.33}$$

If we take $r = 1$ in the MSUERSS method described above, we get the usual single stage UERSS which is already discussed in case (ii) of Section 3.3. In this case, the BLUE $\widehat{\lambda}_2^{n(1)}$ of λ_2 is given by

$$\widehat{\lambda}_2^{n(1)} = \lambda_2^* = \frac{1}{n\psi_{n,n}} \sum_{i=1}^{n} Y_{[n:n]i}$$

with variance given by

$$Var(\widehat{\lambda}_2^{n(1)}) = Var(\lambda_2^*) = \frac{\delta_{n,n,n}}{n\psi_{n,n}^2} \lambda_2^2,$$

where we write $Y_{[n:n]i}$ instead of $Y_{[n:n]i}^{(1)}$ and it represents the measurement of the variable of primary interest of the unit selected in the ranked set sample. Also $\psi_{n,n}$ and $\delta_{n,n,n}$ are obtained by putting $r = 1$ in (3.25) and (3.26), respectively.

Al-Saleh (2004) has considered the steady-state RSS by letting r to $+\infty$. If we apply the steady-state RSS to the above problem, the asymptotic distribution of $Y_{[n:n]i}^{(r)}$ is obtained from (3.22) and is given by the pdf

$$h_{[n:n]i}^{(\infty)}(y) = \frac{\beta_2 \lambda_2^{\beta_2}}{(\lambda_2 + y)^{\beta_2 + 1}} \left\{ 1 - \alpha[2(\frac{\lambda_2}{\lambda_2 + y})^{\beta_2} - 1] \right\}. \quad (3.34)$$

From the definition of MSUERSS, it follows that $Y_{[n:n]i}^{(\infty)}, i = 1, 2, \ldots, n$, are *iid* random variables each with pdf as defined in (3.34). Then $Y_{[n:n]i}^{(\infty)}, i = 1, 2, \ldots, n$, may be regarded as unbalanced steady-state ranked set sample of size n. Then from (3.27) and (3.28), the mean and variance of $Y_{[n:n]i}^{(\infty)}$, for $1 \leq i \leq n$, are obtained as

$$E(Y_{[n:n]i}^{(\infty)}) = \lambda_2 \psi_{n^\infty, n^\infty}$$

and

$$Var(Y_{[n:n]i}^{(\infty)}) = \lambda_2^2 \delta_{n^\infty, n^\infty, n^\infty},$$

where

$$\psi_{n^\infty, n^\infty} = \frac{1}{\beta_2 - 1} - \alpha\beta_2 \left[\frac{2\,\Gamma(2\beta_2 - 1)}{\Gamma(2\beta_2 + 1)} - \frac{\Gamma(\beta_2 - 1)}{\Gamma(\beta_2 + 1)} \right] \quad (3.35)$$

and

$$\delta_{n^\infty,n^\infty,n^\infty} = \frac{2}{(\beta_2-1)(\beta_2-2)} - 2\alpha\beta_2\left(\frac{2\,\boldsymbol{\Gamma}(2\beta_2-2)}{\boldsymbol{\Gamma}(2\beta_2+1)} - \frac{\boldsymbol{\Gamma}(\beta_2-2)}{\boldsymbol{\Gamma}(\beta_2+1)}\right) - \left[\frac{1}{\beta_2-1} - \alpha\beta_2\left(\frac{2\,\boldsymbol{\Gamma}(2\beta_2-1)}{\boldsymbol{\Gamma}(2\beta_2+1)} - \frac{\boldsymbol{\Gamma}(\beta_2-1)}{\boldsymbol{\Gamma}(\beta_2+1)}\right)\right]^2. \tag{3.36}$$

Let $\mathbf{Y}_{[n]}^{(\infty)} = (Y_{[n:n]1}^{(\infty)}, Y_{[n:n]2}^{(\infty)}, \dots, Y_{[n:n]n}^{(\infty)})'$. Then, the BLUE $\widehat{\lambda}_2^{n(\infty)}$ based on $\mathbf{Y}_{[n]}^{(\infty)}$ and the variance of $\widehat{\lambda}_2^{n(\infty)}$ are obtained by taking the limit as $r \to \infty$ in (3.32) and (3.33), respectively, and are given by

$$\widehat{\lambda}_2^{n(\infty)} = \frac{1}{n\psi_{n^\infty,n^\infty}}\sum_{i=1}^{n} Y_{[n:n]i}^{(\infty)}$$

with variance given by

$$Var(\widehat{\lambda}_2^{n(\infty)}) = \frac{\delta_{n^\infty,n^\infty,n^\infty}}{n\psi^2_{n^\infty,n^\infty}}\lambda_2^2, \tag{3.37}$$

where ψ_{n^∞,n^∞} and $\delta_{n^\infty,n^\infty,n^\infty}$ are as given in expressions (3.35) and (3.36), respectively.

From (3.5) and (3.37), we obtain the efficiency $e(\widehat{\lambda}_2^{n(\infty)}/\widetilde{\lambda}_2) = \frac{Var(\widetilde{\lambda}_2)}{Var(\widehat{\lambda}_2^{n(\infty)})}$ of $\widehat{\lambda}_2^{n(\infty)}$ relative to $\widetilde{\lambda}_2$ for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = 0.25(0.25)1$ and are presented in Table 3.

As mentioned earlier, since the concomitant of smallest order statistic possesses the maximum FI about λ_2 when $\alpha < 0$, in this case we consider a multistage lower extreme ranked set sampling (MSLERSS) in which at each stage and from each set we choose a unit of a sample with the smallest value on the auxiliary variable as the units of ranked sets with an objective of exploiting the maximum FI about λ_2 on the ultimately chosen ranked set sample.

Let $Y_{[1:n]i}^{(r)}$, $i = 1, 2, \dots, n$ be the value measured on the variable of primary interest on the units selected at the rth stage of the MSLERSS. Clearly, the distribution of $Y_{[1:n]i}^{(r)}$ is the same as that of $Y_{[1:n^r]}$, the concomitant of the smallest order statistic of n^r *iid* bivariate random variables with FGMBL distribution. Moreover, $Y_{[1:n]i}^{(r)}$, $i = 1, 2, \dots, n$, are also independently distributed with pdf given by

$$h_{[1:n]i}^{(r)}(y) = \frac{\beta_2\lambda_2^{\beta_2}}{(\lambda_2+y)^{\beta_2+1}}\left\{1 + \frac{\alpha(n^r-1)}{n^r+1}[2(\frac{\lambda_2}{\lambda_2+y})^{\beta_2} - 1]\right\}, \tag{3.38}$$

$$y > 0;\ \lambda_2 > 0;\ \beta_2 > 0;\ \alpha < 0.$$

Since both $h^{(r)}_{[1:n]i}(y)$ and $h^{(r)}_{[n:n]i}(y)$ are functions of α, let us rewrite $h^{(r)}_{[1:n]i}(y)$ as $h^{(r)}_{[1:n]i}(y;\alpha)$ and $h^{(r)}_{[n:n]i}(y)$ as $h^{(r)}_{[n:n]i}(y;\alpha)$. Clearly, from (3.22) and (3.38) it can be seen that

$$h^{(r)}_{[1:n]i}(y;\alpha) = h^{(r)}_{[n:n]i}(y;-\alpha). \tag{3.39}$$

Hence, from (3.39), it follows that $E(Y^{(r)}_{[n:n]i})$ for $\alpha > 0$ and $E(Y^{(r)}_{[1:n]i})$ for $\alpha < 0$ are identically equal. Similarly, $Var(Y^{(r)}_{[n:n]i})$ for $\alpha > 0$ and $Var(Y^{(r)}_{[1:n]i})$ for $\alpha < 0$ are identically equal. Consequently, if $\widehat{\lambda}_2^{1(r)}$ is the BLUE of λ_2 involved in the FGMBL distribution for $\alpha < 0$ based on the MSLERSS observations $Y^{(r)}_{[1:n]i}$, $i = 1, 2, \ldots, n$, then the coefficients of $Y^{(r)}_{[1:n]i}$, $i = 1, 2, \ldots, n$ in the BLUE $\widehat{\lambda}_2^{1(r)}$ for $\alpha < 0$ is the same as the coefficients of $Y^{(r)}_{[n:n]i}$, $i = 1, 2, \ldots, n$ in the BLUE $\widehat{\lambda}_2^{n(r)}$ for $\alpha > 0$.

Further, we have $Var(\widehat{\lambda}_2^{1(r)}) = Var(\widehat{\lambda}_2^{n(r)})$ and hence $Var(\widehat{\lambda}_2^{1(1)}) = Var(\widehat{\lambda}_2^{n(1)})$ and $Var(\widehat{\lambda}_2^{1(\infty)}) = Var(\widehat{\lambda}_2^{n(\infty)})$, where $\widehat{\lambda}_2^{1(1)}$ is the BLUE of λ_2 for $\alpha < 0$ based on the usual single stage LERSS observations $Y_{[1:n]i}$, $i = 1, 2, \ldots, n$ and $\widehat{\lambda}_2^{1(\infty)}$ is the BLUE of λ_2 for $\alpha < 0$ based on the unbalanced steady-state RSS observations. The case when $r = 1$ in the MSLERSS method described above reduces to the usual single stage LERSS which is already discussed in case (i) of Section 3.3.

Since $Var(\widehat{\lambda}_2^{1(\infty)})$ for $\alpha < 0$ and $Var(\widehat{\lambda}_2^{n(\infty)})$ for $\alpha > 0$ are the same, values of $e(\widehat{\lambda}_2^{1(\infty)}/\widetilde{\lambda}_2)$ for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = -1, -0.75, -0.50, -0.25$ are the same as that of $e(\widehat{\lambda}_2^{n(\infty)}/\widetilde{\lambda}_2)$ for $n = 2(2)10$, $\beta_2 = 3(1)5$ and $\alpha = 0.25, 0.50, 0.75, 1$ and hence, values of $e(\widehat{\lambda}_2^{1(\infty)}/\widetilde{\lambda}_2)$ for $\alpha = -1, -0.75, -0.50, -0.25$ are not included in Table 3.

From Table 3, it can be seen that efficiency increases as α increases and for a fixed pair (n, α), efficiency increases as β_2 increases. The value of efficiency varies from 1.10890 to 1.63003.

4 Estimation of $1/(\beta_2 - 1)$ using Stokes' RSS when λ_2 is known

In this section, we provide an estimator of $B = \frac{1}{\beta_2 - 1}$ for known values of λ_2. Suppose n sets of sampling units each of size n are drawn from the FGMBL distribution with pdf defined by (1.3), where the scale parameter λ_2 is assumed to be known. As we have seen earlier, since the distribution of $Y_{[r:n]r}$ is the same as that of $Y_{[r:n]}$, from equations (2.2), (2.3) and (3.3) it can be seen that

$$E(Y_{[r:n]r} + Y_{[n-r+1:n]\overline{n-r+1}}) = \frac{2\lambda_2}{\beta_2 - 1}$$

and

$$Var(Y_{[r:n]r} + Y_{[n-r+1:n]\overline{n-r+1}}) = \lambda_2^2 \left\{ \frac{4}{(\beta_2-1)(\beta_2-2)} - \frac{2}{(\beta_2-1)^2} - \frac{2\alpha^2\beta_2^2(n-2r+1)^2}{(n+1)^2} \left(\frac{2\,\Gamma(2\beta_2-1)}{\Gamma(2\beta_2+1)} - \frac{\Gamma(\beta_2-1)}{\Gamma(\beta_2+1)} \right)^2 \right\}.$$

Denote $R_{[r:n]r} = \frac{1}{2}(Y_{[r:n]r} + Y_{[n-r+1:n]\overline{n-r+1}}),\ r = 1, 2, \ldots, [\frac{n}{2}]$, where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Then for the FGMBL distribution, $\frac{R_{[r:n]r}}{\lambda_2}$ is an unbiased estimator of B for each r. Hence an unbiased estimator of B based on Stokes' ranked set sample observations is given by

$$\hat{B} = \frac{1}{[\frac{n}{2}]\lambda_2} \sum_{r=1}^{[\frac{n}{2}]} R_{[r:n]r} \tag{4.1}$$

with variance

$$Var(\hat{B}) = \frac{1}{2([\frac{n}{2}])^2} \sum_{r=1}^{[\frac{n}{2}]} \left\{ \frac{2}{(\beta_2-1)(\beta_2-2)} - \frac{1}{(\beta_2-1)^2} - \frac{\alpha^2\beta_2^2(n-2r+1)^2}{(n+1)^2} \left(\frac{2\,\Gamma(2\beta_2-1)}{\Gamma(2\beta_2+1)} - \frac{\Gamma(\beta_2-1)}{\Gamma(\beta_2+1)} \right)^2 \right\}.$$

Remark 5. If one is very particular of getting an estimate $\hat{\beta}_2$ of β_2, then the concept involved in method of moments may be used to get it from (4.1). For this we write $T = \frac{1}{[\frac{n}{2}]\lambda_2} \sum_{r=1}^{[\frac{n}{2}]} R_{[r:n]r}$ so that $\hat{\beta}_2 = \frac{1+T}{T}$.

5 Concluding remarks

The theory of concomitants of order statistics arising from FGMBL distribution is developed. This development further provides the necessary statistical foundation to formulate RSS strategies for a population random variable following a FGMBL distribution. A modified ranked set sampling strategy called extreme ranked set sampling is further developed. When making measurement on the auxiliary variable is very cheap, one can improve the efficiency of the estimator of the parameter of interest further by

Table 3: The efficiency of $\widehat{\lambda}_2^{n(\infty)}$ relative to $\widetilde{\lambda}_2$ for $n = 2(2)10$, $\beta_2 = 3(1)5$ and for $\alpha = 0.25, 0.50, 0.75, 1$

n	β_2	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 1$
2	3	1.10890	1.23160	1.35923	1.46809
4		1.10893	1.23183	1.36075	1.47562
6		1.10895	1.23215	1.36291	1.48608
8		1.10897	1.23264	1.36623	1.50107
10		1.10901	1.23341	1.37097	1.52028
2	4	1.11779	1.25399	1.40163	1.54084
4		1.11781	1.25421	1.40302	1.54759
6		1.11783	1.25448	1.40502	1.55703
8		1.11786	1.25493	1.40806	1.57071
10		1.11789	1.25565	1.41251	1.58871
2	5	1.12280	1.26686	1.42654	1.58446
4		1.12284	1.26704	1.42785	1.59081
6		1.12284	1.26732	1.42975	1.59971
8		1.12287	1.26774	1.43265	1.61273
10		1.12291	1.26843	1.43693	1.63003

defining multistage ranked set sampling. Accordingly, we have introduced multistage extreme ranked set sampling in FGMBL and analysed its advantages over standard RSS procedures.

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