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Estimation and Reconstruction Based on Left Censored Data from Pareto Model

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Abstract. In this paper, based on a left censored data from the two-parameter Pareto distribution, maximum likelihood and Bayes estimators for the two unknown parameters are obtained. The problem of reconstruction of the past failure times, either point or interval, in the left-censored set-up, is also considered from Bayesian and non-Bayesian approaches. Two numerical examples and a Monte Carlo simulation study are given for illustrative purposes.

Keywords. Bayes estimator, best unbiased reconstructor, conditional median reconstructor, highest conditional density, left censoring, maximum-likelihood reconstructor, reconstruction interval.

MSC: Primary 62G30; Secondary 62N01.

1 Introduction

The two-parameter Pareto distribution (denoted by $P(\alpha, \beta)$) has the cumulative distribution function (cdf)

$$F(x; \alpha, \beta) = 1 - \left(\frac{\beta}{x}\right)^{\alpha}, \quad x \geq \beta > 0, \quad \alpha > 0, \tag{1.1}$$

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and the probability density function (pdf)

$$f(x;\alpha,\beta) = \frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\ , \quad x \geq \beta > 0, \quad \alpha > 0. \tag{1.2}$$

This distribution is called Pareto type-I distribution (see, for example, Johnson et al. (1994)) with parameters α and β.

The Pareto distribution was originated by Pareto (1897) as a model for the distribution of income but is now used as a model in such widely diverse areas as insurance, business, economics, engineering, hydrology and reliability. In addition, the Pareto distribution has been found to provide a good model in biomedical problems, such as survival time following a heart transplant (Bain and Engelhardt, 1992). The origin and other aspects of this distribution can be found in Johnson et al. (1994). Estimations, predictions, and some inference concerning the Pareto distribution were discussed by many authors. Among others, see AL-Hussaini and Ahmad (2003), Madi and Raqab (2004), Ahmadi and Doosparast (2006), Raqab et al. (2007), Soliman (2008) and Raqab et al. (2010).

Censored sampling arises in a life testing experiment whenever the experimenter does not observe (either intentionally or unintentionally) the failure times of all units placed on a life test. The two most common censoring schemes are termed Type-I and Type-II censoring. They can be described as follows. Consider a sample of n units placed on a life-test at time 0. In conventional Type-I censoring, a time T, independent of the failure times, is pre-fixed so that beyond this time no failures will be observed, that is, the experimentation terminates at time T. On the other hand, in the case of conventional Type-II censoring, the number of observed failures is fixed, say m ($m \leq n$), so that at the time of the mth failure, the experimentation terminates, leaving $n-m$ partially observed failure times. One of the drawbacks to the conventional Type-I and Type-II censoring schemes is that they do not allow for removal of units at points other than the terminal point of the experiment. A more general censoring scheme known as progressive censoring scheme, which has this advantage, has become very popular in the last few years. For more details on progressive censoring scheme, see Balakrishnan and Aggarwala (2000).

In this paper, we consider a left censoring scheme which can be described as follows. Let us consider an experiment in which n components are put to test simultaneously at time 0, and the failure times of these components are recorded. Suppose some initial observations are censored possibly due to failures during the time when checks and adjustments are being made on the devices. Suppose, for instance, that out of n items put on life test, the largest $n - r$ life times $X_{(r+1)} < X_{(r+2)} < \cdots < X_{(n)}$ have only been observed and the life times for the first r components remain unobserved or missing. This type of censoring is known as a left censoring scheme. There is a widespread application and use of left censored data in survival analysis and reliability theory. For more details and some applications of left censoring scheme, see Balakrishnan (1989), Balakrishnan and Varadan (1991), Bagger (2005) and Mitra and Kundu (2008).

If $X_{(r+1)} < X_{(r+2)} < \cdots < X_{(n)}$ is the available left censored sample from a random sample of size n from a life-time distribution with cdf $F(x;\theta)$ and pdf $f(x;\theta)$. Then, the joint pdf of $\mathbf{X} = (X_{(r+1)}, X_{(r+2)}, \cdots, X_{(n)})$ is given by

$$f(\mathbf{x};\theta) = \frac{n!}{r!}[F(x_{(r+1)};\theta)]^r \prod_{i=r+1}^{n} f(x_{(i)};\theta), \quad x_{(r+1)} < x_{(r+2)} < \cdots < x_{(n)}, \tag{1.3}$$

where $\mathbf{x} = (x_{(r+1)}, x_{(r+2)}, \cdots, x_{(n)})$ is the vector of observations.

Reconstruction of the past failure times in the left censored setup is an interesting topic, especially in the viewpoint of actuarial, medical and engineering sciences. Suppose that we observe the largest $n - r$ life times $X_{(r+1)} < X_{(r+2)} < \cdots < X_{(n)}$ from Pareto distribution. Our main aim is to reconstruct the past failure times $X_{(1)} < X_{(2)} < \cdots < X_{(r)}$ based on the $n - r$ observed order statistics $\mathbf{X} = (X_{(r+1)}, \cdots, X_{(n)})$. We study this problem via non-Bayesian and Bayesian approaches and present several reconstructors of $Y = X_{(s)}, (1 \le s \le r)$.

In the recent years, several authors have considered reconstruction problems involving order statistics. Balakrishnan et al. (2009) have addressed the problem of reconstructing past records from the known values of future records when the underlying distributions were exponential and Pareto distributions. Razmkhah *et. al.* (2010) have derived point and interval reconstructors for the missing order statistics from two parameter exponential distribution. Recently, Asgharzadeh et al. (2012) have discussed reconstructors of times to failure of units censored in a left-censored sample from the proportional reversed hazard rate models.

In this paper, various reconstructors of times to failure of units censored in a left-censored sample from the Pareto distribution are discussed. The paper is organized as follows. Section 2 presents the maximum likelihood and Bayes estimators for the two unknown parameters. In Section 3, different point reconstructors are derived for the past failure times. We also provide two types of reconstruction intervals (RI's) for the past failure times. Finally in Section 4, the analysis of two data sets and a Monte Carlo simulation study are presented for illustrative purposes.

2 Estimation of the Parameters

Let $X_{(r+1)} < X_{(r+2)} < \cdots < X_{(n)}$ be the available left censored sample from a random sample of size n from a Pareto distribution, with cdf and pdf given, respectively, by equations (1.1) and (1.2). In this section, we consider the problem of estimation with both Bayesian and non-Bayesian approaches for the unknown parameters α and β. From equations (1.1), (1.2) and (1.3), the joint pdf of $\mathbf{X} = (X_{(r+1)}, X_{(r+2)}, \cdots, X_{(n)})$ is given by

$$\begin{aligned} f(\mathbf{x};\alpha,\beta) &= \frac{n!}{r!}\alpha^{n-r}\beta^{\alpha(n-r)}\left[1-\left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]^{r} \\ &\quad \times(\prod_{i=r+1}^{n} x_{(i)})^{-\alpha-1}, \quad \beta \le x_{(r+1)}, \ \alpha > 0. \qquad (2.1) \end{aligned}$$

From (2.1), we note that $(\sum_{i=r+1}^{n} \ln X_{(i)}, X_{(r+1)})$ is a jointly sufficient statistic for (α, β).

2.1 Maximum Likelihood Estimation

The log-likelihood function is given by

$$\begin{aligned} \ln L(\mathbf{x};\alpha,\beta) &\propto (n-r)\ln\alpha + \alpha(n-r)\ln\beta + r\ln\left[1-\left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right] \\ &\quad -(\alpha+1)\sum_{i=r+1}^{n}\ln x_{(i)}. \qquad (2.2) \end{aligned}$$

From (2.2), we drive the log-likelihood equation for β as

$$\frac{\partial \ln L(\mathbf{x},\alpha,\beta)}{\partial\beta} = \frac{\alpha(n-r)}{\beta} - \frac{\alpha r\left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}}{\beta\left[1-\left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]} = 0. \qquad (2.3)$$

Thus, we obtain the maximum likelihood estimator (MLE) of β as a function of α as

$$\hat{\beta}(\alpha) = \left(\frac{n-r}{n}\right)^{1/\alpha} X_{(r+1)}.$$

Substituting $\hat{\beta}(\alpha)$ in (2.2), we obtain the profile log-likelihood of α and we can obtain the MLE of α by maximizing the profile log-likelihood of α with respect to α. The profile log-likelihood equation for α is

$$\frac{\partial \ln L(\mathbf{x}, \alpha, \beta)}{\partial \alpha} = \frac{n-r}{\alpha} - \sum_{i=r+1}^{n} \ln \frac{x_{(i)}}{x_{(r+1)}} = 0. \qquad (2.4).$$

From (2.4), we obtain the MLE of α as

$$\hat{\alpha} = \frac{(n-r)}{\sum_{i=r+1}^{n} \ln \frac{X_{(i)}}{X_{(r+1)}}}. \qquad (2.5)$$

Now, the MLE of β can be obtained as $\hat{\beta}(\widehat{\alpha})$.

2.2 Bayes Estimation

Under the assumption that both of the parameters of α and β are unknown, we can consider the following joint prior density function

$$\pi(\alpha, \beta) \propto \alpha^{a} \beta^{\alpha b - 1} c^{-\alpha}, \qquad \alpha > 0,\ 0 < \beta < d, \qquad (2.6)$$

for α and β, where a, b, c, d are positive constants and $d^b < c$. This prior was first suggested by Lwin (1972) and later generalized by Arnold and Press (1983, 1989). Such a prior specifies $\pi(\alpha)$ as a gamma distribution with parameters a and $\log c - b \log d$ and $\pi(\beta|\alpha)$ as a power function distribution of the form

$$\pi(\beta|\alpha) \propto b\ \alpha\ \beta^{b\alpha - 1} d^{-b\alpha}, \quad 0 < \beta < d.$$

Note that the noninformative prior

$$\pi(\alpha, \beta) \propto \frac{1}{\alpha\beta}, \qquad \alpha > 0,\ \beta > 0,$$

is specified by letting $a = -1, c = 1, b = 0$ and $d \to \infty$.

Now, the posterior density function of α and β given the data, denoted by $\pi(\alpha, \beta \mid \underline{x})$, can be obtained using (2.1) and (2.6) as

$$\pi(\alpha, \beta \mid \underline{x}) = \frac{1}{R(\underline{x})} \alpha^{a+n-r} \beta^{\alpha(b+n-r)-1} (c \prod_{i=r+1}^{n} x_{(i)})^{-\alpha}$$

$$\times \left[1 - \left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]^{r},\ \alpha > 0,\ 0 < \beta < M, \qquad (2.7)$$

where $M = \min(d, x_{(r+1)})$ and

$$\begin{aligned} R(\underline{x}) &= \Gamma(a+n-r)\sum_{j=0}^{r}\frac{\binom{r}{j}(-1)^j}{(n-r+j+b)} \\ &\quad\times\left[\ln\left(cx_{(r+1)}^{j}M^{r-b-n-j}\prod_{i=r+1}^{n}x_{(i)}\right)\right]^{(r-a-n)}. \end{aligned}$$

Therefore, the Bayes estimator of α under a squared error loss (SEL) is

$$\begin{aligned} \widehat{\alpha}_B &= E(\alpha \mid \underline{x}) \\ &= \int_0^\infty\int_0^M \frac{1}{R(\underline{x})}\alpha^{a+n-r+1}\beta^{\alpha(b+n-r)-1}c^{-\alpha} \\ &\quad\times\left[1-\left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]^r\prod_{i=r+1}^{n}x_{(i)}^{-(\alpha+1)}d\beta d\alpha \\ &= \frac{1}{R(\underline{x})}\Gamma(a+n-r+1)\sum_{j=0}^{r}\frac{\binom{r}{j}(-1)^j}{(n-r+j+b)} \\ &\quad\times\left[\ln\left(cx_{(r+1)}^{j}M^{r-b-n-j}\prod_{i=r+1}^{n}x_{(i)}\right)\right]^{(r-a-n-1)}. \end{aligned} \tag{2.8}$$

Similarly, the Bayes estimator of β can be obtained as

$$\begin{aligned} \widehat{\beta}_B &= E(\beta \mid \underline{x}) \\ &= \int_0^\infty\int_0^M \frac{1}{R(\underline{x})}\alpha^{a+n-r}\beta^{\alpha(b+n-r)}c^{-\alpha} \\ &\quad\times\left[1-\left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]^r\prod_{i=r+1}^{n}x_{(i)}^{-(\alpha+1)}d\beta d\alpha \\ &= \frac{M}{R(\underline{x})}\sum_{j=0}^{r}\binom{r}{j}(-1)^j[S_j(\underline{x})]^{r-a-n} \\ &\quad\times\int_0^\infty\frac{1}{((n-r+b+j)z+S_j(\underline{x}))}z^{a+n-r}e^{-z}dz, \end{aligned}$$

where

$$S_j(\underline{x}) = \ln c + j\ln x_{(r+1)} - (n-r+b+j)\ln M + \sum_{i=r+1}^{n}\ln x_{(i)}.$$

Note that the Bayes estimator of β can be written as

$$\widehat{\beta}_B = \frac{M}{R(\underline{x})} \sum_{j=0}^{r} \frac{\binom{r}{j}(-1)^j [S_j(\underline{x})]^{r-a-n}}{n-r+b+j} \psi\left(a+n-r, \frac{S_j(\underline{x})}{n-r+b+j}\right), \tag{2.9}$$

where

$$\psi(x,y) = \int_0^\infty \frac{t^x}{t+y} e^{-t} dt.$$

A partial tabulation of $\psi(x,y)$ is provided by Arnold and Press (1989).

3 Reconstruction of the Past Failure Times

Suppose $\mathbf{X} = (X_{(r+1)}, ..., X_{(n)})$ be the $n-r$ left censored sample from a population with pdf $f(x;\theta)$ with parameter $\theta \in R^k$. Our objective is to reconstruct the sth past failure time $Y = X_{(s)}$ $(1 \le s \le r)$ based on the observed censored sample $\mathbf{X} = (X_{(r+1)}, \cdots, X_{(n)})$.

Due to Markovian property of order statistics, the conditional distribution of $Y = X_{(s)}$, given $\mathbf{X} = (x_{(r+1)}, \cdots, x_{(n)})$ is equal to the conditional distribution of Y given $X_{(r+1)} = x_{(r+1)}$ which is given by

$$\begin{aligned} &f(y|x_{(r+1)};\theta) \\ &\quad = s\binom{r}{s} f(y;\theta)\,[F(y;\theta)]^{s-1} \left[F(x_{(r+1);\theta}) - F(y;\theta)\right]^{r-s} \left[F(x_{(r+1);\theta})\right]^{-r}, \\ &\quad y \le x_{(r+1)}. \end{aligned} \tag{3.1}$$

For the Pareto distribution with cdf and pdf given, respectively, in (1.1) and (1.2), (3.1) becomes

$$\begin{aligned} f(y|x_{(r+1)};\alpha,\beta) &= s\binom{r}{s}\frac{\alpha}{\beta}\left(\frac{\beta}{y}\right)^{\alpha+1}\left[1-\left(\frac{\beta}{y}\right)^{\alpha}\right]^{s-1} \\ &\times \left[\left(\frac{\beta}{y}\right)^{\alpha} - \left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]^{r-s} \\ &\times \left[1-\left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]^{-r}, \beta \le y \le x_{(r+1)}. \end{aligned} \tag{3.2}$$

In this section, we obtain several reconstructors of $Y = X_{(s)}$ $(1 \le s \le r)$, either point or interval, on the basis of $\mathbf{X} = (x_{(r+1)}, \cdots, x_{(n)})$ via classical and Bayesian approaches.

3.1 Likelihood Reconstruction Method

In likelihood reconstruction method, the principle of maximum likelihood is applied to the joint reconstruction and estimation of a past random variable and an unknown parameter. We assume dependence between present and past, and the approach is non-Bayesian. Let $\mathbf{X} = (X_{(r+1)}, ..., X_{(n)})$ and $Y = X_{(s)}$ $(1 \leq s \leq r)$ have the joint pdf $f(x, y; \theta)$ indexed by the parameter $\theta \in R^k$. The problem here will be to reconstruct the past random variable Y , having observed $\mathbf{X}$. Thus, viewed as a function of y and θ, we define

$$L(y, \theta; \mathbf{x}) = f(\mathbf{x}, y; \theta)$$

to be the reconstructive likelihood function (RLF) of y and θ. Note that the RLF can be rewritten as

$$L(y, \theta; \mathbf{x}) = f(y|\mathbf{x}; \theta) f(\mathbf{x}; \theta).$$

Suppose $\hat{Y}_p = u(\mathbf{X})$ and $\hat{\theta} = v(\mathbf{X})$ are statistics for which

$$L(u(\mathbf{x}), v(\mathbf{x}); \mathbf{x}) = sup_{(y,\theta)} L(y, \theta; \mathbf{x}),$$

then $u(\mathbf{X})$ is said to be the maximum likelihood reconstructor (MLR) of Y and $v(\mathbf{X})$ the reconstructive maximum likelihood estimator (RMLE) of θ.

For the Pareto distribution, using (2.1) and (3.2), the RLF of Y and $\theta = (\alpha, \beta)$ can be written as

$$\begin{aligned} L(y, \alpha, \beta; \mathbf{x}) &= f(y|x_{(r+1)}; \alpha, \beta) f(\mathbf{x}; \alpha, \beta) \\ &= c \left(\frac{\alpha}{\beta}\right)^{n-r+1} \left(\frac{\beta}{y}\right)^{\alpha+1} \left[1 - \left(\frac{\beta}{y}\right)^{\alpha}\right]^{s-1} \\ &\times \left[\left(\frac{\beta}{y}\right)^{\alpha} - \left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right]^{r-s} \\ &\times \prod_{i=r+1}^{n} \left(\frac{\beta}{x_{(i)}}\right)^{\alpha+1} \qquad , \beta \leq y \leq x_{(r+1)}, \quad (3.3) \end{aligned}$$

where c denotes a constant factor.

The reconstructive log-likelihood function of Y and (α, β) is

$$\begin{aligned}\ln L(y, \alpha, \beta; \mathbf{x}) \propto\ & (n-r+1)\ln(\frac{\alpha}{\beta}) + (\alpha+1)\ln(\frac{\beta}{y}) \\ +\ & (s-1)\ln\left[1-\left(\frac{\beta}{y}\right)^{\alpha}\right] \\ +\ & (r-s)\ln\left[\left(\frac{\beta}{y}\right)^{\alpha} - \left(\frac{\beta}{x_{(r+1)}}\right)^{\alpha}\right] \\ +\ & \sum_{j=r+1}^{n}(\alpha+1)\ln(\frac{\beta}{x_{(i)}}). \end{aligned} \tag{3.4}$$

From (3.4), the reconstructive likelihood equation (RLE) for β is given by

$$\frac{\partial \ln L(y, \alpha, \beta)}{\partial \beta} = \frac{\alpha(n-s+1)}{\beta} - \frac{\alpha(s-1)(\frac{\beta}{y})^{\alpha}}{\beta\left(1-(\frac{\beta}{y})^{\alpha}\right)} = 0. \tag{3.5}$$

From (3.5), we obtain the RMLE of β as a function of y and α, say $\hat{\beta}(y, \alpha)$, as

$$\hat{\beta}(y, \alpha) = \left(\frac{n-s+1}{n}\right)^{\frac{1}{\alpha}} y. \tag{3.6}$$

Substituting $\hat{\beta}(y, \alpha)$ in (3.4), the MLR of Y (say $\hat{y}_{MLR}$) and the RMLE of α (say $\hat{\alpha}_{RML}$) can be obtained by maximizing the profile log-likelihood $\ln L(y, \alpha, \hat{\beta}(y, \alpha); \mathbf{x})$ with respect to y and α, respectively. The reconstructive likelihood equations (RLEs) for $Y = X_{(s)}$ $(1 \leq s \leq r)$ and α are given by

$$\frac{\partial \ln L(y, \alpha, \hat{\beta}(y, \alpha); \mathbf{x})}{\partial y} = \left(\frac{y}{x_{(r+1)}}\right)^{-\alpha} - \frac{\alpha(n-s)-1}{\alpha(n-r)-1} = 0, \tag{3.7}$$

$$\begin{aligned}\frac{\partial \ln L(y, \alpha, \hat{\beta}(y, \alpha); \mathbf{x})}{\partial \alpha} =\ & \frac{n-r+1}{\alpha} - \frac{(r-s)\left(\frac{y}{x_{(r+1)}}\right)^{\alpha}\ln(\frac{y}{x_{(r+1)}})}{1-\left(\frac{y}{x_{(r+1)}}\right)^{\alpha}} \\ +\ & \sum_{i=r+1}^{n}\ln(\frac{y}{x_{(i)}}) = 0. \end{aligned} \tag{3.8}$$

Equations (3.7) and (3.8) do not yield explicit solutions for y and α and hence must be solved numerically to obtain $\hat{y}_{MLR}$ and $\hat{\alpha}_{RML}$. Once

$\hat{y}_{MLR}$ and $\hat{\alpha}_{RML}$ are obtained from (3.7) and (3.8), the RMLE of β, say $\hat{\beta}_{RML}$, can be obtained from (3.6) as

$$\hat{\beta}_{RML} = \hat{\beta}(\hat{y}_{MLR}, \hat{\alpha}_{RML}).$$

For a special case, when $s = r$, it is easy to show that

$$\begin{aligned}
\hat{y}_{MLR} &= x_{(r+1)}, \\
\hat{\alpha}_{RML} &= \frac{n-r+1}{\sum_{i=r+1}^{n} \ln(\frac{x_{(i)}}{x_{(r+1)}})}, \\
\hat{\beta}_{RML} &= \left(\frac{n-r+1}{n}\right)^{1/\hat{\alpha}_{RML}} \hat{y}_{MLR}.
\end{aligned}$$

3.2 Conditional Reconstruction Method

In conditional reconstruction method, the conditional distribution of $Y = X_{(s)}$ $(1 \leq s \leq r)$ given $\mathbf{X} = (X_{(r+1)}, \cdots, X_{(n)})$ is applied to derive point and interval reconstructors of Y.

A statistic $\hat{Y}$ which is used to reconstruct $Y = X_{(s)}$ is called a best unbiased reconstructor (BUR) of Y, if $E(\hat{Y}) = E(Y)$ and its reconstructor error variance $\mathrm{Var}(\hat{Y} - Y)$ is less than or equal to that of any other unbiased reconstructor of Y. Since the conditional distribution of Y given $\mathbf{X} = (X_{(r+1)}, \cdots, X_{(n)})$ is just the distribution of Y given $X_{(r+1)}$, therefore the BUR of Y is

$$\hat{Y}_{BUR} = E(Y|X_{(r+1)}).$$

By (3.2), we have

$$\begin{aligned}
\widehat{Y}_{BUR} &= \int_{\beta}^{x_{(r+1)}} y f(y|x_{(r+1)}, \alpha, \beta) dy \\
&= \int_0^1 \beta \left[1 - u\left(1 - (\frac{\beta}{x_{r+1}})^{\alpha}\right)\right]^{-\frac{1}{\alpha}} \frac{u^{s-1}(1-u)^{r-s}}{Beta(s, r-s+1)} du. \quad (3.9)
\end{aligned}$$

If the parameters α and β are unknown, they have to be estimated in this integral. Thus one would replace α and β by their corresponding MLEs and obtain an approximate BUR of Y.

The median Y given $\mathbf{X}$ is called the conditional median reconstructor (CMR). So, a reconstructor $\hat{Y}$ is called the CMR of Y, if we have

$$P_{\alpha,\beta}(Y \leq \hat{Y}|\mathbf{X} = \mathbf{x}) = P_{\alpha,\beta}(Y \geq \hat{Y}|\mathbf{X} = \mathbf{x}).$$

Using the relation

$$
\begin{aligned}
&P_{\alpha,\beta}(Y \geq \hat{Y}|X_{(r+1)} = x_{(r+1)}) \\
&\quad = P_{\alpha,\beta}\left(\frac{1-(\frac{\beta}{Y})^{\alpha}}{1-(\frac{\beta}{x_{(r+1)}})^{\alpha}} \geq \frac{1-(\frac{\beta}{\hat{Y}})^{\alpha}}{1-(\frac{\beta}{x_{(r+1)}})^{\alpha}}|X_{(r+1)} = x_{(r+1)}\right),
\end{aligned}
$$

and by the fact that ,

$$
\left(\frac{1-(\frac{\beta}{Y})^{\alpha}}{1-(\frac{\beta}{x_{(r+1)}})^{\alpha}}\right) \mid X_{(r+1)} = x_{(r+1)} \sim Beta(s, r-s+1), \tag{3.10}
$$

we obtain the CMR of Y as

$$
\hat{Y}_{CMR} = \beta\left[1 - Med(U)\left(1-(\frac{\beta}{x_{(r+1)}})^{\alpha}\right)\right]^{-\frac{1}{\alpha}}, \tag{3.11}
$$

where U is a beta random variable with parameters s and $r-s+1$. If α and β are unknown, as recommended by Balakrishnan et al. (2009), one method is substituting α and β by their corresponding MLEs.

For the special case $s = r$, we have $Med(U) = \sqrt[r]{1/2}$, and hence we obtain

$$
\hat{Y}_{CMR} = \beta\left[1-(\frac{1}{2})^{\frac{1}{r}}\left(1-(\frac{\beta}{x_{(r+1)}})^{\alpha}\right)\right]^{-\frac{1}{\alpha}}.
$$

Now, we consider two approaches to obtain reconstruction intervals (RIs) for $Y = X_{(s)}$ $(1 \leq s \leq r)$ based on the left censored sample $\mathbf{X} = (X_{(r+1)}, X_{(r+2)}, ..., X_{(n)})$. Let us take the random variable Z as

$$
Z = \left(\frac{1-(\frac{\beta}{Y})^{\alpha}}{1-(\frac{\beta}{x_{(r+1)}})^{\alpha}}\right).
$$

As mentioned in (3.10), Z given $X_{(r+1)} = x_{(r+1)}$ has a $Beta(s, r-s+1)$ distribution with pdf

$$
g(z|x_{(r+1)}) = s\binom{r}{s} z^{s-1}(1-z)^{r-s}, \qquad 0 < z < 1.
$$

So, we can consider Z as a pivotal quantity to obtain the reconstruction interval for Y. By noting that the $Beta(s, r-s+1)$ distribution is unimodal, then a $100(1-\gamma)\%$ two sided RI for Y is

$$
\left(\beta\left[1-b_{1-\frac{\gamma}{2}}\left(1-(\frac{\beta}{x_{(r+1)}})^{\alpha}\right)\right]^{-\frac{1}{\alpha}}, \quad \beta\left[1-b_{\frac{\gamma}{2}}\left(1-(\frac{\beta}{x_{(r+1)}})^{\alpha}\right)\right]^{-\frac{1}{\alpha}}\right), \tag{3.12}
$$

where b_γ stands for 100γth percentile of $Beta(s, r-s+1)$. When α and β are unknown, we can replace them by their corresponding MLEs.

Now let us consider another reconstruction interval for $Y = X_{(s)}$. By substituting α and β in (3.2) by their MLEs, we can obtain the approximate density of Y given $X_{(r+1)} = x_{(r+1)}$ as

$$\begin{aligned}
\widehat{f}(y|x_{(r+1)}) &= f(y|x_{(r+1)}, \hat{\alpha}, \hat{\beta}) \\
&= s\binom{r}{s}\frac{\hat{\alpha}}{\hat{\beta}}\left(\frac{\hat{\beta}}{y}\right)^{\hat{\alpha}+1}\left[1-\left(\frac{\hat{\beta}}{y}\right)^{\hat{\alpha}}\right]^{s-1} \\
&\times \left[\left(\frac{\hat{\beta}}{y}\right)^{\hat{\alpha}} - \left(\frac{\hat{\beta}}{x_{(r+1)}}\right)^{\hat{\alpha}}\right]^{r-s} \\
&\times \left[1-\left(\frac{\hat{\beta}}{x_{(r+1)}}\right)^{\hat{\alpha}}\right]^{-r}, \; y \le x_{(r+1)}. \qquad (3.13)
\end{aligned}$$

We can easily observe that this conditional density is a unimodal function of

$$\widehat{Z} = \left(\frac{1-(\frac{\hat{\beta}}{Y})^{\hat{\alpha}}}{1-(\frac{\hat{\beta}}{x_{(r+1)}})^{\hat{\alpha}}}\right),$$

for $s > 1$ and $r > s$ (i.e; $s = 2, ..., r-1$). Then, the $100(1-\gamma)\%$ highest conditional density (HCD) RI for Y is

$$\left(\hat{\beta}\left[1-w_1\left(1-(\frac{\hat{\beta}}{x_{(r+1)}})^{\hat{\alpha}}\right)\right]^{-\frac{1}{\hat{\alpha}}}, \quad \hat{\beta}\left[1-w_2\left(1-(\frac{\hat{\beta}}{x_{(r+1)}})^{\hat{\alpha}}\right)\right]^{-\frac{1}{\hat{\alpha}}}\right),$$

where w_1 and w_2 are the simultaneous solution of the following equations:

$$1-\gamma = \int_{w_1}^{w_2} g(z|x_{(r+1)})dz, \qquad (3.14)$$

and

$$g(w_1|x_{(r+1)}) = g(w_2|x_{(r+1)}). \qquad (3.15)$$

Now, we simplify the equations (3.14) and (3.15) as

$$B_{w_2}(s, r-s+1) - B_{w_1}(s, r-s+1) = 1-\gamma, \qquad (3.16)$$

and

$$\left(\frac{1-w_2}{1-w_1}\right)^{r-s} = \left(\frac{w_1}{w_2}\right)^{s-1}, \qquad (3.17)$$

where

$$B_t(a,b) = \frac{1}{B(a,b)} \int_0^t x^{a-1}(1-x)^{b-1}\, dx,$$

is the incomplete beta function.

Note that for special case $s = r$, we find a simple expression for $100(1-\gamma)\%$ HCD RI for Y, which is

$$\left(\hat{\beta}\left[1-\gamma\left(1-(\frac{\hat{\beta}}{x_{(r+1)}})^{\hat{\alpha}}\right)\right]^{-\frac{1}{\hat{\alpha}}}, x_{(r+1)} \right).$$

3.3 Bayesian Reconstruction Method

Suppose that we have observed the $n-r$ left censored sample $\mathbf{X} = (X_{(r+1)}, \cdots, X_{(n)})$ from a Pareto distribution with pdf (1.2). In this section, on the basis of such a sample, we consider the problem of reconstruction $Y = X_{(s)}$ $(s < r)$ based on Bayesian approach. The Bayes reconstructive density function of $Y = X_{(s)}$ given $X_{(r+1)} = x_{(r+1)}$ is given by

$$f^*(y \mid x_{(r+1)}) = \int_0^\infty \int_0^M f(y \mid x_{(r+1)}, \alpha, \beta)\pi(\alpha, \beta \mid \underline{x})d\beta d\alpha. \tag{3.18}$$

By substituting (2.7) and (3.2) into (3.18) and applying the binomial expansion, we get

$$\begin{aligned} f^*(y \mid x_{(r+1)}) &= \int_0^\infty \int_0^M \frac{s\binom{r}{s}}{R(\underline{x})} \alpha^{a+n-r+1} \beta^{\alpha(b+n-s+1)-1} c^{-\alpha} \\ &\times \left[1-\left(\frac{\beta}{y}\right)^\alpha\right]^{s-1} \prod_{i=r+1}^{n} x_{(i)}^{-\alpha} \\ &\times \left(y^{-\alpha} - x_{(r+1)}^{-\alpha}\right)^{r-s} y^{-(\alpha+1)} d\beta d\alpha \\ &= \frac{s\binom{r}{s}}{R(\underline{x})} \sum_{k=0}^{s-1} \binom{s-1}{k} (-1)^k \int_0^\infty \alpha^{a+n-r+1} c^{-\alpha} \prod_{i=r+1}^{n} x_{(i)}^{-\alpha} \\ &\times \left(y^{-\alpha} - x_{r+1}^{-\alpha}\right)^{r-s} y^{-(\alpha(k+1)+1)} \\ &\times \int_0^M \beta^{\alpha(n-s+b+1+k)-1} d\beta d\alpha \end{aligned}$$

$$
\begin{aligned}
&= \frac{s\binom{r}{s}}{R(\underline{x})}\sum_{k=0}^{s-1}\sum_{l=0}^{r-s}\binom{s-1}{k}\binom{r-s}{l}\frac{(-1)^{k+r-s-l}}{(n-s+b+1+k)} \\
&\quad \times\int_0^{\infty}\alpha^{a+n-r}c^{-\alpha}\prod_{i=r+1}^{n}x_{(i)}^{-\alpha} \\
&\quad \times y^{-(\alpha(k+l+1)+1)}\; x_{r+1}^{-\alpha(r-s-l)}\; M^{\alpha(n-s+b+1+k)}d\alpha \\
&= s\binom{r}{s}\frac{\Gamma(a+n-r+1)}{yR(\underline{x})}\sum_{k=0}^{s-1}\sum_{l=0}^{r-s}\frac{\binom{s-1}{k}\binom{r-s}{l}(-1)^{k+l}}{(n-s+b+1+k)} \\
&\quad \times\left[\ln\left(c\prod_{i=r+1}^{n}x_{(i)}\; x_{(r+1)}^{r-s-l}\; y^{k+l+1}\; M^{s-b-n-k-1}\right)\right]^{(r-a-n-1)}
\end{aligned}
$$

where $\beta < y \leq x_{(r+1)}$.

Bayesian prediction bounds for Y are obtained by evaluating

$$P(Y \geq \lambda \mid x_{(r+1)}) = \int_{\lambda}^{x_{(r+1)}} f^*(y \mid x_{(r+1)})dy,$$

for some positive λ. The probability $P(Y \geq \lambda \mid x_{(r+1)})$ can be shown to be

$$
\begin{aligned}
&s\binom{r}{s}\frac{\Gamma(a+n-r+1)}{(a-n-r)R(\underline{x})}\sum_{k=0}^{s-1}\sum_{l=0}^{r-s}\binom{s-1}{k}\binom{r-s}{l} \\
&\times\frac{(-1)^{k+l-1}}{(n+k-s+b+1)(k+l+1)} \\
&\times\left[\left(\ln(c\prod_{i=r+1}^{n}x_{(i)}\; M^{s-b-n-k-1}\; x_{(r+1)}^{r+k-s+1})\right)^{r-a-n}\right. \\
&\left.-\left(\ln(c\prod_{i=r+1}^{n}x_{(i)}\; M^{s-b-n-k-1}\; x_{(r+1)}^{r-s-l}\;\lambda^{k+l+1})\right)^{(r-a-n)}\right] \qquad (3.19)
\end{aligned}
$$

Now, the $100(1-\gamma)\%$ Bayesian reconstruction interval for $Y = X_{(s)}$ is given by $(L_B(x_{(r+1)}), U_B(x_{(r+1)}))$ where $L_B(x_{(r+1)})$ and $U_B(x_{(r+1)})$ are the lower and upper reconstruction bounds, respectively, satisfying

$$P[Y \geq L_B(x_{(r+1)}) \mid x_{(r+1)}] = 1 - \frac{\gamma}{2},$$

and

$$P[Y \geq U_B(x_{(r+1)}) \mid x_{(r+1)}] = \frac{\gamma}{2}.$$

Numerical methods are required to obtain the lower and upper $100(1-\gamma)\%$ reconstruction bounds for Y.

Note that Bayesian point reconstructors can be also obtained from $f^*(y \mid x_{(r+1)})$, the Bayes reconstructive density function of $Y = X_{(s)}$ given $X_{(r+1)} = x_{(r+1)}$, and the given loss function. Under a absolute error loss (AEL) function , the Bayesian point reconstructor of $Y = X_{(s)}$ (denoted by $\widehat{Y}_{AEL}$) is the median of the Bayes reconstructive function $f^*(y \mid x_{(r+1)})$. That is a number m satisfies the relation

$$P(Y \geq m \mid x_{(r+1)}) = \int_m^{x_{(r+1)}} f^*(y \mid x_{(r+1)})dy = \frac{1}{2},$$

which is obtained numerically using (3.19).

Under the squared error loss (SEL) function, the Bayesian point reconstructor of $Y = X_{(s)}$ is the mean of the Bayes reconstructive function $f^*(y \mid x_{(r+1)})$. It is

$$\begin{aligned}
\widehat{Y}_{SEL} &= \int_\beta^{x_{(r+1)}} y f^*(y \mid x_{(r+1)})dy \\
&= s\binom{r}{s}\frac{\Gamma(a+n-r+1)}{R(\underline{x})}\sum_{k=0}^{s-1}\sum_{l=0}^{r-s}\frac{\binom{s-1}{k}\binom{r-s}{l}(-1)^{k+l}}{(n-s+b+1+k)} \\
&\times \int_\beta^{x_{(r+1)}}\left[\ln\left(c\prod_{i=r+1}^{n} x_{(i)}\, x_{(r+1)}^{r-s-l}\, y^{k+l+1}\, M^{s-b-n-k-1}\right)\right]^{(r-a-n-1)} dy.
\end{aligned}$$

As we can see, the Bayesian point reconstructor under SEL function is difficult to obtain. Moreover, since the parameter β is unknown, it has to be estimated in this integral.

4 Numerical Examples and Simulations

In this section, two numerical examples are given to illustrate the reconstruction methods proposed in this paper. We apply the proposed methods to one of simulated data set and another practical data set. Further, a Monte Carlo simulation is conducted to compare the performance of the point reconstructors as well as the reconstruction intervals (RIs). The performance of the point reconstructors are based on their biases and mean square reconstruction errors (MSREs) while the performance of the RIs are based on their average lengths and their coverage probabilities. All of the computations are performed using the Mathematical Package Maple 13.

4.1 Example 1 (Simulated Data):

The following ordered data has been generated from the Pareto distribution (1.1) with parameters $\alpha = 2.5$ and $\beta = 1$.

```
1.0139  1.0149  1.0445  1.0712  1.0715  1.0869  1.1045  1.1875
1.2185  1.3485  1.3734  1.3780  1.4685  1.7171  1.7698  1.9566
2.0276  2.1614  2.2861  2.7478
```

We assumed that here the parameters of of prior density in (2.6) are as $a = 3, b = 1, c = 4$ and $d = 2$. For this example, we suppose that the first ($r = 5$) observations are not observed. The MLEs of the parameters α and β are obtained as $\hat{\alpha} = 2.621$ and $\hat{\beta} = 0.9739$. Using (2.8) and (2.9), the Bayes estimates of α and β are computed as $\hat{\alpha_B} = 2.562$ and $\hat{\beta_B} = 0.9577$. The different point reconstructors MLR, BUR, CMR and Bayesian point reconstructor (BPR) of $Y = X_{(s)}$ $(s \leq 5)$ and also the 95% RI's are displayed in Table 1. The BPR is obtained using AEL function.

Table 1. The values of point reconstructors and 95% RIs for the simulated data.

	Exact value	MLR	BUR	CMR	BPR	Pivotal method	HCD method	Bayesian method
$X_{(1)}$	1.0139	0.8422	0.9902	0.9862	1.0822	(0.9743, 1.0272)	(—,—)	(.146180, 1.08517)
$X_{(2)}$	1.0149	0.8959	1.0070	1.0050	1.0730	(0.9788, 1.0500)	(0.9763, 1.0444)	(.229425, 1.07296)
$X_{(3)}$	1.0445	0.9543	1.0260	1.0250	1.0868	(0.9878, 1.0672)	(0.9878, 1.0672)	(.146209, 1.08686)
$X_{(4)}$	1.0712	1.0180	1.0440	1.0460	1.0868	(1.0016, 1.0796)	(1.0064, 1.0832)	(.146224, 1.08686)
$X_{(5)}$	1.0715	1.0870	1.0650	1.0690	1.0868	(1.0223, 1.0861)	(0.9785,1.0868)	(.229465, 1.08686)

4.2 Example 2 (Real Data):

Here we analyze one real data set to illustrate the reconstruction procedure proposed in Section 3. The following data represent the time to breakdown of a type of electronic insulating material subject to a constant-voltage stress. These data are taken from Nelson (1970) and has been used earlier by Tiku and Akkaya (2004).

```
0.35  0.59  0.96  0.99  1.69  1.97  2.07  2.58  2.71  2.90  3.67
3.99  5.35  13.77  25.50
```

We checked the validity of the Pareto model based on the parameters $\alpha = 0.51$ and $\beta = 0.35$. We used the Kolmogorov-Smirnov (K-S) test for this data set. It is observed that the K-S distance between the fitted and the empirical distribution functions, and the corresponding p-value are respectively

$$K - S = 0.2869, \qquad \text{and} \qquad p - value = 0.1393.$$

So, the fit of Pareto distribution to the above data set is reasonable.

In order to analyze these data, we took $r = 4$, and for $s = 1, 2, 3, 4$ we reconstructed $X_{(s)}$ $(s \leq 4)$ using different methods discussed in Sections 3. The different point reconstructors of $Y = X_{(s)}$ $(s \leq 4)$ and also the 95% RI's are displayed in Table 2. To compute the Bayes reconstructors, since we do not have any prior information, we assumed that the prior is noninformative, *i.e.*, $a = -1, b = 0, c = 1, d = \infty$.

Table 2. The values of point reconstructors and 95% RIs for the real data.

	Exact value	MLR	BUR	CMR	BPR	Pivotal method	HCD method	Bayesian method
$X_{(1)}$	0.35	0.2402	0.4564	0.4191	1.6803	(0.3524, 0.7732)	(—,—)	(0.1173, 1.6878)
$X_{(2)}$	0.59	0.4838	0.6091	0.5597	1.6242	(0.3770, 1.1098)	(0.3672, 1.0405)	(0.1173, 1.6246)
$X_{(3)}$	0.96	1.4930	0.8326	0.7884	1.5765	(0.4371, 1.4446)	(0.4554, 1.5244)	(0.1174, 1.5767)
$X_{(4)}$	0.99	1.6900	1.1050	1.1900	1.5404	(0.5689, 1.6645)	(0.3697,1.6900)	(0.1174,1.5407)

From Tables 1 and 2, we observe that the BUR and CMR work well. They are close to the realized observation. We observe that the RIs obtained by pivotal method are close to those provided by the HCD method. Bayesian method does not work well to obtain the Bayesian point reconstructor and the 95% RIs. The RIs intervals obtained by Bayesian method are wider than the other RIs. It is also observed that all of the RIs proposed contain the realized observation.

4.3 Simulations

Since the performance of the different methods cannot be compared theoretically, we present here a Monte Carlo simulation study to compare the performances of the different reconstructors proposed in the previous sections. We compare the performances of the point reconstructors MLR, BUR and CMR in terms of their biases and mean square reconstruction errors (MSREs). We also compare the performances of the RIs obtained by using pivotal and HCD methods in terms of the average confidence lengths, and coverage probabilities. In this simulation, we randomly generated 1000 left-censored sample $X_{(r+1)}, X_{(r+2)}, \cdots, X_{(n)}$ from $P(\alpha, \beta)$ distribution for $n = 20, 30$ with $(\alpha, \beta) = (0.5, 1), (2.5, 0.5), (2.5, 1)$. We then obtained the different point reconstructors MLR, BUR, and CMR for the sth past failure time $Y = X_{(s)}$, where $1 \leq s \leq r$. We also computed the 95% RIs for $Y = X_{(s)}$ by using the results given in Section 3. All the computations are performed using Visual Maple (V13) package.

For various choices of r and s, Tables 3 and 5 present the biases and MSREs of point reconstructors obtained from this simulation study. Table 3 is for $n = 20$ and Table 5 is for $n = 30$. The average confidence lengths and the corresponding coverage probabilities of 95% RIs are also reported in Tables 4 and 6.

From Tables 3 and 5, we observe that the BUR and the CMR are the best point reconstructors. They compare very well with the MLR. The likelihood reconstruction method does not work very well. It provides the highest biases and MSREs. It is also observed that for fixed r, the biases and MSREs are decreasing as s is increasing, which is reasonable as we move away from available censored sample. For better understanding, the average MSREs of the different reconstructors are presented in Figures 1 and 2 for different sample sizes. From Figures 1 and 2, it is observed that for fixed r and s, the MSREs are decreasing as n is increasing.

Now we compare the different RIs obtained by pivotal and HCD methods. From Tables 4 and 6, we observe that the HCD RIs work better than the pivotal RIs in terms of confidence lengths in most of the cases considered. Also, the confidence lengths decrease as n increases. It should be mentioned here that unlike HCD method, the pivotal method is computationally easy.

Table 3. Biases and MSREs of point reconstructions for $n = 20$.

parameter	r	s	MLR	BUR	CMR
$\alpha = 2.5, \beta = 1$	1	1	0.02200(0.001408)	0.001437(0.0008989)	0.01181(0.001055)
	2	1	-0.009757(0.001604)	0.01886(0.001818)	0.01715(0.001757)
		2	-0.03722(0.002984)	-0.07280(0.006806)	-0.05002(0.004073)
	3	1	-0.02987(0.003223)	0.03741(0.003672)	0.03450(0.003468)
		2	0.0002708(0.002603)	0.02690(0.003223)	0.02685(0.003222)
	4	1	-0.07380(0.008874)	0.03159(0.004357)	0.02792(0.004153)
		2	-0.05291(0.006053)	0.01454(0.003366)	0.01278(0.003321)
		3	-0.05478(0.006004)	-0.02955(0.003740)	-0.02813(0.003661)
		4	0.04920(0.005501)	-0.01504(0.002982)	0.03271(0.004070)
	5	1	-0.09642(0.01354)	0.04889(0.006663)	0.04475(0.006301)
		2	-0.05611(0.007466)	0.05340(0.007111)	0.05067(0.006843)
		3	-0.01064(0.004693)	0.05760(0.007706)	0.05677(0.007613)
		4	0.04533(0.006626)	0.07000(0.009263)	0.07271(0.009662)
		5	-0.01372(0.004389)	-0.09262(0.01230)	-0.03181(0.005138)
$\alpha = 0.5, \beta = 1$	1	1	-0.08667(0.05366)	-0.1577(0.06598)	-0.1460(0.06325)
	2	1	0.3111(0.1877)	0.1998(0.1105)	0.1882(0.1050)
		2	0.285(0.1622)	0.1665(0.09622)	0.2028(0.1134)
	4	1	0.5839(0.6938)	0.3016(0.2851)	0.2770(0.2655)
		2	0.6181(0.6748)	0.3651(0.3162)	0.3497(0.3022)
		3	0.4456(0.4243)	0.2374(0.2221)	0.2391(0.2235)
		4	0.6941(0.7614)	0.4733(0.4393)	0.5674(0.5635)
$\alpha = 2.5, \beta = 0.5$	1	1	0.4881(0.2410)	0.006327(0.0002951)	0.01152(0.0003930
	2	1	0.4624(0.2283)	0.01493(0.0006206)	0.01408(0.0005960)
		2	0.4727(0.2325)	0.01018(0.0004560)	0.02153(0.0008313)
	3	1	0.3802(0.1904)	0.007309(0.0006374)	0.005850(0.0006202)
		2	0.4055(0.1979)	0.02085(0.001024)	0.02083(0.001023)
	4	1	0.01605(0.01751)	0.01785(0.001174)	0.01603(0.001115)
		2	0.2300(0.1226)	-0.02381(0.001311)	-0.02469(0.001354)
		3	0.3087(0.1459)	0.01202(0.001026)	0.01272(0.001045)
		4	0.3496(0.1564)	-0.004858(0.0007253)	0.01896(0.001124)
	5	1	-0.09069(0.04633)	0.02058(0.001517)	0.01850(0.001440)
		2	-0.06154(0.01639)	0.01329(0.001247)	0.01191(0.001216)
		3	0.04995(0.06838)	-0.04766(0.003340)	-0.04806(0.003380)
		4	0.2179(0.1037)	0.03189(0.002086)	0.03327(0.002178)
		5	0.2675(0.1120)	0.005421(0.0009476)	0.03578(0.002302)

Table 4. Means of the lengths and coverage probability of 95% RIs for $n = 20$.

parameter	r	s	Pivotal method	HCD method
$\alpha = 2.5, \beta = 1$	1	1	0.01904(0.5600)	–(–)
	2	1	0.03432(0.4860)	–(–)
		2	0.03476(0.7450)	–(–)
	3	1	0.04359(0.4150)	–(–)
		2	0.05223(0.6760)	0.05223(0.6760)
	4	1	0.05042(0.3930)	–(–)
		2	0.06489(0.5710)	0.06346(0.5600)
		3	0.06723(0.7410)	0.06694(0.7390)
		4	0.05737(0.8620)	–(–)
	5	1	0.05450(0.3610)	–(–)
		2	0.07282(0.5050)	0.06966(0.4820)
		3	0.08317(0.6480)	0.08318(0.6480)
		4	0.08014(0.7520)	0.07895(0.7570)
		5	0.06635(0.8310)	0.08198(0.3570)
$\alpha = 0.5, \beta = 1$	1	1	0.1085(0.5610)	–(–)
	2	1	0.2046(0.5100)	–(–)
		2	0.2132(0.7500)	–(–)
	4	1	0.3211(0.4200)	–(–)
		2	0.4292(0.5870)	0.4155(0.5790)
		3	0.4577(0.7090)	0.4603(0.7060)
		4	0.4076(0.8360)	–(–)
$\alpha = 2.5, \beta = 0.5$	1	1	0.009599(0.5720)	–(–)
	2	1	0.01696(0.4590)	–(–)
		2	0.01730(0.7600)	–(–)
	3	1	0.02195(0.4690)	–(–)
		2	0.02612(0.6450)	0.02612(0.6450)
	4	1	0.02512(0.3630)	–(–)
		3	0.03232(0.5710)	0.03161(0.5600)
		3	0.03371(0.7150)	0.03356(0.7260)
		4	0.02847(0.8430)	–(–)
	5	1	0.02730(0.3400)	–(–)
		2	0.03640(0.5220)	0.03482(0.5110)
		3	0.04071(0.6520)	0.04071(0.6520)
		4	0.04087(0.7680)	0.04026(0.7770)
		5	0.03282(0.8690)	0.04057,(0.3290)

Table 5. Biases and MSREs of point reconstructions for $n = 30$.

parameter	r	s	MLR	BUR	CMR
$\alpha = 2.5, \beta = 1$	1	1	0.02203(0.0009280)	0.004957(0.0004561)	0.01522(0.0006718)
	2	1	0.003465(0.0006477)	0.02147(0.001092)	0.02041(0.001049)
		2	0.002959(0.0006416)	-0.02708(0.001333)	-0.005297(0.0006523)
	3	1	-0.02083(0.001422)	0.02078(0.001393)	0.01890(0.001320)
		2	-0.003523(0.0009923)	0.01230(0.001116)	0.01246(0.001121)
	4	1	-0.04086(0.002910)	0.02515(0.001841)	0.02283(0.001732)
		2	-0.02529(0.001889)	0.01543(0.001441)	0.01439(0.001409)
		3	0.01166(0.001390)	0.02643(0.001910)	0.02763(0.001976)
		4	0.03177(0.002226)	-0.02283(0.001626)	0.02186(0.001680)
	5	1	-0.06146(0.005423)	0.02779(0.002382)	0.02510(0.002245)
		2	-0.04204(0.003378)	0.02409(0.002174)	0.02246(0.002101)
		3	0.003327(0.001620)	0.04430(0.003524)	0.04397(0.003493)
		4	0.006315(0.001667)	0.01956(0.001944)	0.02183(0.002043)
		5	0.02782(0.002267)	-0.03911(0.002843)	0.01714(0.001755)
$\alpha = 0.5, \beta = 1$	1	1	0.1231(0.03086)	0.07409(0.02001)	0.08535(0.02210)
	2	1	0.1866(0.06006)	0.1213(0.03633)	0.1147(0.03458)
		2	0.02481(0.02528)	-0.05370(0.02465)	-0.02349(0.02343)
	3	1	0.2300(0.09870)	0.1226(0.04972)	0.1114(0.04658)
		2	0.1723(0.07519)	0.08275(0.04485)	0.08124(0.04455)
	4	1	0.2982(0.1551)	0.1486(0.06718)	0.1347(0.06247)
		2	0.2392(0.1241)	0.1026(0.05970)	0.09501(0.05775)
		3	0.2819(0.1452)	0.1718(0.08280)	0.1748(0.08404)
		4	0.3394(0.1828)	0.2014(0.09620)	0.2728(0.1362)
	5	1	0.3039(0.1851)	0.1034(0.06662)	0.08749(0.06249)
		2	-0.02344(0.09528)	-0.2121(0.1102)	-0.2237(0.1145)
		3	0.3560(0.2203)	0.1883(0.1025)	0.1839(0.1005)
		4	0.07220(0.09753)	-0.06283(0.07762)	-0.05589(0.07748)
		5	0.2821(0.1753)	0.1115(0.08828)	0.2062(0.1289)
$\alpha = 2.5, \beta = 0.5$	1	1	0.4943(0.2445)	0.0001875(0.0001032)	0.005317(0.0001335)
	2	1	0.4918(0.2439)	0.008570(0.0002464)	0.008040(0.0002378)
		2	0.4896(0.2402)	-0.003201(0.0001575)	0.007686(0.0002125)
	3	1	0.4863(0.2418)	0.01059(0.0003623)	0.009653(0.0003435)
		2	0.4666(0.2204)	-0.009566(0.0003034)	-0.009488(0.0003021)
	4	1	0.4438(0.2136)	-0.008255(0.0003617)	-0.009426(0.0003832)
		2	0.4574(0.2232)	0.007949(0.0003732)	0.007435(0.0003654)
		4	0.4635(0.2244)	-0.002791(0.0002806)	0.01952(0.0006772)
	5	1	0.3908(0.1955)	0.004442(0.0004301)	0.003115(0.0004211)
		2	0.4053(0.2010)	0.008676(0.0004951)	0.007842(0.0004817)
		3	0.4249(0.2041)	0.003783(0.0003600)	0.003620(0.0003589)
		4	0.4220(0.2002)	0.009945(0.0004841)	0.01107(0.0005087)
		5	0.4145(0.1891)	-0.02574(0.001022)	0.002426(0.0004053)

Table 6. Means of the lengths and coverage probability of 95% RIs for $n = 30$.

parameter	r	s	Pivotal method	HCD method
$\alpha = 2.5, \beta = 1$	1	1	0.01280(0.5760)	−(−)
	2	1	0.02229(0.4930)	−(−)
		2	0.02275(0.7680)	−(−)
	3	1	0.02896(0.4450)	−(−)
		2	0.03412(0.6770)	0.03412(0.6770)
	4	1	0.03259(0.4160)	−(−)
		2	0.04229(0.6350)	0.04149(0.6290)
		3	0.04242(0.7520)	0.04210(0.7490)
		4	0.03541(0.8410)	−(−)
	5	1	0.03601(0.3950)	−(−)
		2	0.04717(0.5370)	0.04536(0.5240)
		3	0.05191(0.6950)	0.05192(0.6950)
		4	0.05060(0.7920)	0.04957(0.7890)
		5	0.04042(0.8860)	0.05241(0.3250)
$\alpha = .5, \beta = 1$	1	1	0.06999(0.5760)	−(−)
	2	1	0.1246(0.4930)	−(−)
		2	0.1291(0.7680)	−(−)
	3	1	0.1652(0.4450)	−(−)
		2	0.1982(0.6770)	0.1982(0.6770)
	4	1	0.1880(0.4160)	−(−)
		2	0.2504(0.6350)	0.2440(0.6290)
		3	0.2591(0.7520)	0.2588(0.7490)
		4	0.2262(0.8590)	−(−)
	5	1	0.2064(0.3900)	−(−)
		2	0.2858(0.5660)	0.2719(0.5450)
		3	0.3265(0.6880)	0.3266(0.6880)
		4	0.3231(0.7860)	0.3198(0.7810)
		5	0.2712(0.8910)	0.3199(0.3030)
$\alpha = 2.5, \beta = .5$	1	1	0.006382(0.5760)	−(−)
	2	1	0.01137(0.5230)	−(−)
		2	0.01132(0.7830)	−(−)
	3	1	0.01441(0.4580)	−(−)
		2	0.01718(0.6790)	0.01718(0.6790)
	4	1	0.01646(0.4230)	−(−)
		3	0.02103(0.6100)	0.02063(0.6020)
		4	0.01764(0.8670)	−(−)
	5	1	0.01781(0.3900)	−(−)
		2	0.02390(0.5340)	0.02298(0.5250)
		3	0.02599(0.7120)	0.02599(0.7120)
		4	0.02493(0.7760)	0.02443(0.7750)
		5	0.02036(0.8860)	0.02639(0.3130)

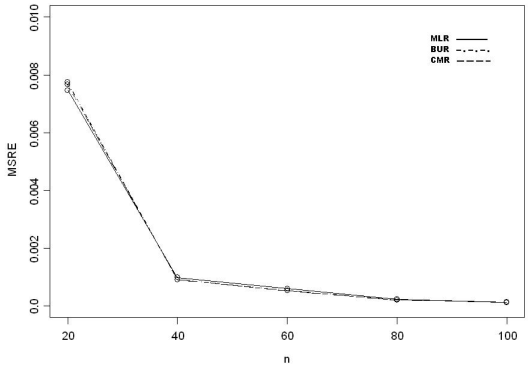

Figure 1: MSREs of the different reconstructors for different sample size n when $r = 5, s = 3$.

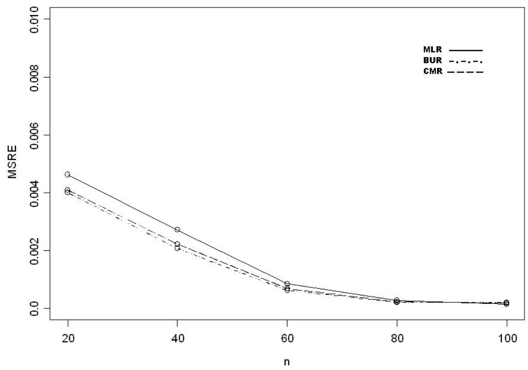

Figure 2: MSREs of the different reconstructors for different sample size n when $r = 5, s = 4$

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