

## On Orthogonalization Approach to Construct a Multiple Input Transfer Function Model

Mahnaz Khalafi, Majid Azimmohseni, Mohammad Kordkatuli

Department of Statistics, Faculty of Science, Golestan University, Gorgan, Iran.

**Abstract.** In this article, a special type of orthogonalization is obtained to construct a multiple input transfer function model. By using this technique, construction of a transfer function model is divided to sequential construction of transfer function models with less input time series. Furthermore, based on real and simulated time series we provide an instruction to adequately perform the stages of orthogonalization algorithm.

**Keywords.** Frequency response function, frequency transformation, impulse response weight, orthogonalization, transfer function model.

**MSC:** 62M10, 62M15.

### 1 Introduction

Suppose  $\{(y_t, x_t)', t \in \mathbb{Z}\}$  is a jointly weakly stationary time series. Then, a transfer function model in terms of input time series  $x_t$  and output time series  $y_t$  is given by

$$y_t = \sum_{j=0}^{\infty} \nu_j x_{t-j} + \eta_t = \nu(B)x_t + \eta_t, \quad (1)$$

---

Mahnaz Khalafi(✉)(m.khalafi@gu.ac.ir), Majid Azimmohseni(m.azim@gu.ac.ir),  
Mohammad Kordkatuli(mk.volcano66@yahoo.com)

Received: December 2013; Accepted: September 2014

where  $\eta_t$  is second order stationary time series and assumed to be independent of input time series  $x_t$ , Box and Jenkins (1970). The coefficients  $\nu_0, \nu_1, \dots$  are called impulse response weights, and  $\nu(B) = \sum_{j=0}^{\infty} \nu_j B^j$  where  $B$  stands for backward shift operator. Model (1) has two fundamental requirements, namely causality and stability.

Model (1) is said to be causal if the value of  $y_t$  at the present time is affected only by the values of  $x_t$  at the current and past time. For notational convenience, let us use " $y_t C x_t$ " to call causality of transfer function model  $y_t$  on  $x_t$ . On the other hand, model (1) is said to be stable if the sequence of impulse response functions is absolutely summable, i.e.  $\sum_{j=0}^{\infty} |\nu_j| < \infty$ . Since, an infinity of parameters are included in model (1), it is desirable to use the parsimonious form of this model, i.e.

$$y_t = \frac{\omega(B)}{\delta(B)} B^b x_t + \eta_t, \quad (2)$$

where  $\omega(B) = \omega_0 - \omega_1 B - \omega_2 B^2 - \dots - \omega_s B^s$  and  $\delta(B) = 1 - \delta_1 B - \delta_2 B^2 - \dots - \delta_r B^r$ . The stability of model (2) holds when all the roots of  $\delta(B)$  lie outside of unit circle. Also, positivity of parameter  $b$  guarantees the causality of this form of transfer function model. Parameter  $b$  identifies the lag in which the output and input time series have the most significant correlation. Note that by comparing models (1) and (2), we can obtain the relation  $\nu(B) = \frac{\omega(B)}{\delta(B)} B^b$ . Indeed this relation includes the following set of linear equations between parameters of these models:

$$\begin{cases} \nu_h = 0 & h < b \\ \nu_h = \delta_1 \nu_{h-1} + \delta_2 \nu_{h-2} + \dots + \delta_r \nu_{h-r} + \omega_0 & h = b \\ \nu_h = \delta_1 \nu_{h-1} + \delta_2 \nu_{h-2} + \dots + \delta_r \nu_{h-r} - \omega_{h-b} & h = b + 1, \dots, b + s \\ \nu_h = \delta_1 \nu_{h-1} + \delta_2 \nu_{h-2} + \dots + \delta_r \nu_{h-r} & h > b + s. \end{cases} \quad (3)$$

The frequency transformation of model (1) is of crucial important in this study. Define the frequency response function  $H(\lambda)$  as Fourier transform of impulse response weights, i.e.

$$H(\lambda) = \sum_{j=0}^{\infty} \nu_j e^{-i\lambda j},$$

then frequency transformations of model (1) can be obtained as

$$f_{xy}(\lambda) = H(\lambda) f_{xx}(\lambda), \quad (4)$$

and

$$f_{yy}(\lambda) = |H(\lambda)|^2 f_{xx}(\lambda) + f_{\eta\eta}(\lambda), \quad (5)$$

(Priestley, 1982), where  $f_{xy}(\lambda)$  is cross-spectral density and  $f_{xx}(\lambda)$  and  $f_{yy}(\lambda)$  are marginal spectral densities.

Generally, a transfer function model with  $m$  input time series is given by

$$y_t = \sum_{k=1}^m \nu_k(B)x_{k,t} + \eta_t = \sum_{k=1}^m \sum_{j=0}^{\infty} \nu_{kj}x_{k,t-j} + \eta_t. \tag{6}$$

The parsimonious model relating to (6) has the following representation:

$$y_t = \sum_{k=1}^m \frac{\omega_k(B)}{\delta_k(B)} B^{b_k} x_{k,t} + \eta_t. \tag{7}$$

The causality and stability of model (6) and (7) hold whenever their separated models with one input time series admire conditions of models (1) and (2) respectively.

The frequency transformation of model (6) is given by

$$f_{x_k y} = H_1(\lambda)f_{x_k x_1} + H_2(\lambda)f_{x_k x_2}(\lambda) + \dots + H_m(\lambda)f_{x_k x_m}(\lambda), \quad k = 1, \dots, m, \tag{8}$$

where  $H_k(\lambda) = \sum_{j=0}^{\infty} \nu_{kj}e^{-i\lambda j}$ ,  $k = 1, \dots, m$ . Specially for  $m = 2$ , the system of linear equations (8) reduces to:

$$\begin{cases} f_{x_1 y}(\lambda) = H_1(\lambda)f_{x_1 x_1}(\lambda) + H_2(\lambda)f_{x_1 x_2}(\lambda) \\ f_{x_2 y}(\lambda) = H_1(\lambda)f_{x_2 x_1}(\lambda) + H_2(\lambda)f_{x_2 x_2}(\lambda). \end{cases} \tag{9}$$

In this article, we propose a special type of orthogonalization approach to construct a multiple input transfer function model. There are several achievements to construct a transfer function model both in time domain and frequency domain. In their fundamental work, Box and Jenkins (1970) studied the construction of single input transfer function models using time and frequency domain techniques. Also, they proposed a frequency domain method for constructing transfer function models with two input time series. Later, Pukkila (1982) extended the frequency domain approach to construct multiple input transfer function models. There are also several proposed methods in time domain to construct a transfer function model. Liu and Hanssen (1982) applied AR polynomials as filtering tools to reduce the correlation among input time series and transform a multiple input transfer function model to separated models with one input time series. Edlund (1989) proposed a two-step biased regression strategy to construct a transfer function model. Muller and Wei (1997) obtained an iterative least squares

method to estimate parameters of a transfer function model.

The book by Wei (2006) cited some common methods to construct a transfer function model both in time domain and frequency domain.

Recent developments in constructing a transfer function model are directed toward finding more flexible transfer function models such as nonlinear models in the works Cai et al. (2000) and Fan and Yao (2003), transfer function models with time-varying coefficients in the works Zho and Chon (2004), Bibi (2006) and Moura et al. (2012), and transfer functions with non-stationary time series such as locally stationary processes in the work Nason et al. (2000).

Orthogonalization is one of the common techniques for dimension reduction in model building. Dimension reduction for a model can be made to achieve to different purposes including reduction in bias of parameters, simplifying the possible computational operations and obtaining more accurate forecasts. Dimension reduction in time series models either linear or nonlinear has been also used for the same purposes. To receive general information on this issue, one may refer to the works of Park et al. (2009, 2010). Specifically, Park (2011) applied an orthogonalization approach for reduction in dimension of a single input transfer function model.

One particular purpose of dimension reduction is transforming a complex model to several models with relatively simpler structure. In fact, by applying a dimension reduction plan, a model building divides into step-by-step approach. Our main objective in this manuscript is finding a dimension reduction algorithm to construct a multiple input transfer function model.

In spite of significant achievements in constructing a multiple input transfer function, complexities still exist even in the linear transfer function model (6). The complexity of constructing a transfer function model with several input time series is mainly due to the correlation among its input time series. If input time series are highly correlated, cross-correlation can be seriously misleading. The main strategy to overcome this obstacle has been reducing or removing the correlation among input time series by some filtering approaches such as prewhitening. Unfortunately, by using these filtering techniques, the information contained in relation among input time series is completely ignored. Specially, those applicable relations in practice that can be established by time-lagged regression models.

By utilizing the orthogonalization approach, we can obtain a transfer function model with orthogonal input time series, without any loss

of information. Besides, because of simplicity of orthogonalization algorithm provided in this paper, we recommend employing it in applied fields.

The rest of the paper is organized as follows. In section 2, we present an algorithm and its theoretical backgrounds to perform the orthogonalization. Using this technique, we conduct a simulation study to construct a multiple input time series in section 3. In section 4, we use the orthogonalization approach to fit a transfer function model to a real hydrological data set.

## 2 Orthogonalization Algorithm

Orthogonalization is a common technique in theory of linear model to produce orthogonal explanatory variables. Consequently, by orthogonalizing explanatory variables of a linear model, the original model transforms to more than one linear models with less explanatory variables. In this section, we show that this approach can be adequately used to construct a transfer function model with several input time series.

First, we present this method for a transfer function model with two input time series and then extend it to a multiple input transfer function model.

### 2.1 Orthogonalization Algorithm for a Transfer Function Model with Two Input Time Series

Let a transfer function model with two input time series be as follows

$$y_t = \nu_1(B)x_{1,t} + \nu_2(B)x_{2,t} + \eta_t. \tag{10}$$

In model (10), without loss of generality, we assume that " $x_{2,t} C x_{1,t}$ ". According to the system of linear equations (9), it is plain to see that

$$H_2(\lambda) = \frac{f_{x_2y}(\lambda)f_{x_1x_1}(\lambda) - f_{x_1y}(\lambda)f_{x_2x_1}(\lambda)}{f_{x_2x_2}(\lambda)f_{x_1x_1}(\lambda) - f_{x_1x_2}(\lambda)f_{x_2x_1}(\lambda)} \tag{11}$$

and

$$H_1(\lambda) = \frac{f_{x_1y}(\lambda) - H_2(\lambda)f_{x_1x_2}(\lambda)}{f_{x_1x_1}(\lambda)}. \tag{12}$$

The following algorithm describes the orthogonalization approach in detail for constructing the transfer function model given in (10).

Step 1. Fit a transfer function model between two inputs

$$x_{2,t} = \nu_{x_1x_2}(B)x_{1,t} + \eta_{1,t} \tag{13}$$

and then compute the residual time series, i.e.

$$\eta_{1,t} = x_{2,t} - \nu_{x_1x_2}(B)x_{1,t}. \quad (14)$$

To fit a single input transfer function model we will present a detailed instruction in sections 3 and 4.

The frequency transformation related to (13) and (14) are given respectively by

$$f_{x_1x_2}(\lambda) = H_{x_1x_2}(\lambda)f_{x_1x_1}(\lambda), \quad (15)$$

$$f_{\eta_1\eta_1}(\lambda) = f_{x_2x_2}(\lambda) - \frac{|f_{x_1x_2}(\lambda)|^2}{f_{x_1x_1}(\lambda)}, \quad (16)$$

(Priestley, 1982). By replacing  $x_{2,t}$  in the model (10) by  $\eta_{1,t}$  we can obtain a transfer function model with orthogonal input time series. Because of orthogonality, the frequency transformation of the exchanged model reduces to

$$f_{x_1y}(\lambda) = H_{x_1y}(\lambda)f_{x_1x_1}(\lambda), \quad (17)$$

$$f_{\eta_1y}(\lambda) = H_{\eta_1y}(\lambda)f_{\eta_1\eta_1}(\lambda). \quad (18)$$

The relations (17) and (18) indicate that a transfer function model with two or more orthogonal input time series can be decomposed into separated transfer function models with one input time series without loss of any information. Therefore, it leads to the step 2.

Step 2. Fit separately the transfer function models

$$y_t = \nu_{x_1y}(B)x_{1,t} + \eta_{2,t} \quad \text{and} \quad y_t = \nu_{\eta_1y}(B)\eta_{1,t} + \eta_{3,t}. \quad (19)$$

The frequency transformation of these models are given respectively in (17) and (18). On the other hand, from (14) we can obtain

$$f_{\eta_1y}(\lambda) = f_{x_2y}(\lambda) - \frac{f_{x_2x_1}(\lambda)}{f_{x_1x_1}(\lambda)}f_{x_1y}(\lambda), \quad (20)$$

(Priestley, 1982). By replacing (16) and (20) into (18),  $H_{\eta_1y}$  can be written as

$$\begin{aligned} H_{\eta_1y}(\lambda) &= \frac{f_{\eta_1y}(\lambda)}{f_{\eta_1\eta_1}(\lambda)} \\ &= \frac{f_{x_2y}(\lambda)f_{x_1x_1}(\lambda) - f_{x_1y}(\lambda)f_{x_2x_1}(\lambda)}{f_{x_2x_2}(\lambda)f_{x_1x_1}(\lambda) - f_{x_1x_2}(\lambda)f_{x_2x_1}(\lambda)} \\ &= H_2(\lambda), \end{aligned} \quad (21)$$

where  $H_2(\lambda)$  is given by (11). The equation (21) yields that  $\nu_{\eta_1 y}(B)$  operates identically as  $\nu_2(B)$  in (10). Also, from (12) it is plain to see that

$$H_1(\lambda) = H_{x_1 y}(\lambda) - H_{\eta_1 y}(\lambda)H_{x_1 x_2}(\lambda). \quad (22)$$

According to the properties of Fourier transform, we can obtain that

$$\nu_1(B) = (\nu_{x_1 y} - (\nu_{\eta_1 y} * \nu_{x_1 x_2}))(B),$$

where " \* " is used for convolution. Therefore, it leads to the last step. Step 3. Derive  $\nu_1(B)$  and  $\nu_2(B)$  in (10) from the following relations:

$$\nu_1(B) = (\nu_{x_1 y} - (\nu_{\eta_1 y} * \nu_{x_1 x_2}))(B),$$

$$\nu_2(B) = \nu_{\eta_1 y}(B).$$

## 2.2 Orthogonalization Algorithm for a Transfer Function Model with Several Input Time Series

Assume a multiple input transfer function model with the following form

$$y_t = \sum_{k=1}^m \nu_k(B)x_{k,t} + \eta_t. \quad (23)$$

By splitting input time series into two groups, model (23) can be restated as

$$y_t = \boldsymbol{\nu}'_1(B)\mathbf{x}_{1,t} + \boldsymbol{\nu}'_2(B)\mathbf{x}_{2,t} + \eta_t, \quad (24)$$

where  $\boldsymbol{\nu}_1(B) = (\nu_1(B), \dots, \nu_p(B))'$ ,  $\boldsymbol{\nu}_2(B) = (\nu_{p+1}(B), \dots, \nu_m(B))'$ ,  $\mathbf{x}_{1,t} = (x_{1,t}, \dots, x_{p,t})'$  and  $\mathbf{x}_{2,t} = (x_{p+1,t}, \dots, x_{m,t})'$ . Concerning to frequency transformation of model (24), define  $\mathbf{F}_{\mathbf{x}_i \mathbf{x}_i}(\lambda)$ ,  $i = 1, 2$  as marginal spectral density matrices of input time series and the matrix  $\mathbf{F}_{\mathbf{x}_1 \mathbf{x}_2}(\lambda)$  as their cross-spectral density matrix. Also, let  $\mathbf{f}_{\mathbf{x}_i y}(\lambda)$ ,  $i = 1, 2$  be cross-spectral density vectors of the output and input time series. In fact, model (24) is an analogues of model (10) in which its frequency transformations can be given by

$$\begin{cases} \mathbf{f}_{\mathbf{x}_1 y}(\lambda) = \mathbf{F}_{\mathbf{x}_1 \mathbf{x}_1}(\lambda)\mathbf{h}_1(\lambda) + \mathbf{F}_{\mathbf{x}_1 \mathbf{x}_2}(\lambda)\mathbf{h}_2(\lambda), \\ \mathbf{f}_{\mathbf{x}_2 y}(\lambda) = \mathbf{F}_{\mathbf{x}_2 \mathbf{x}_1}(\lambda)\mathbf{h}_1(\lambda) + \mathbf{F}_{\mathbf{x}_2 \mathbf{x}_2}(\lambda)\mathbf{h}_2(\lambda). \end{cases} \quad (25)$$

where  $\mathbf{h}_1(\lambda) = (H_1(\lambda), \dots, H_p(\lambda))'$  and  $\mathbf{h}_2(\lambda) = (H_{p+1}(\lambda), \dots, H_m(\lambda))'$  are frequency response vectors relating to the groups of input time series

respectively.

According to system of equations (25), we can derive

$$\mathbf{h}_2(\lambda) = [\mathbf{F}_{\mathbf{x}_2\mathbf{x}_2}(\lambda) - \mathbf{F}_{\mathbf{x}_2\mathbf{x}_1}(\lambda)\mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}^{-1}(\lambda)\mathbf{F}_{\mathbf{x}_1\mathbf{x}_2}(\lambda)]^{-1} [\mathbf{f}_{\mathbf{x}_2\mathbf{y}}(\lambda) - \mathbf{F}_{\mathbf{x}_2\mathbf{x}_1}(\lambda)\mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}^{-1}(\lambda)\mathbf{f}_{\mathbf{x}_1\mathbf{y}}(\lambda)] \quad (26)$$

and

$$\mathbf{h}_1(\lambda) = \mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}^{-1}(\lambda)\mathbf{f}_{\mathbf{x}_1\mathbf{y}}(\lambda) - \mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}^{-1}(\lambda)\mathbf{F}_{\mathbf{x}_1\mathbf{x}_2}(\lambda)\mathbf{h}_2(\lambda). \quad (27)$$

In order to investigate the orthogonalization approach, at the first we assume hypothetically that  $\mathbf{x}_{2,t}$  and  $\mathbf{x}_{1,t}$  are inherently orthogonal. Therefore, the system of linear equations (25) reduces to

$$\begin{cases} \mathbf{f}_{\mathbf{x}_1\mathbf{y}}(\lambda) = \mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}(\lambda)\mathbf{h}_1(\lambda), \\ \mathbf{f}_{\mathbf{x}_2\mathbf{y}}(\lambda) = \mathbf{F}_{\mathbf{x}_2\mathbf{x}_2}(\lambda)\mathbf{h}_2(\lambda). \end{cases} \quad (28)$$

The relation (28) reveals that for two orthogonal groups of input time series, model (24) can be split into two transfer function models with less input time series, i.e.

$$y_t = \boldsymbol{\nu}'_1(B)\mathbf{x}_{1,t} + \varepsilon_{1,t} \quad \text{and} \quad y_t = \boldsymbol{\nu}'_2(B)\mathbf{x}_{2,t} + \varepsilon_{2,t}. \quad (29)$$

However, for a multiple input transfer function model, several types of causality relations may happen among input time series. As a special case, suppose that " $\mathbf{x}_{2,t} \perp \mathbf{x}_{1,t}$ ". Then, orthogonalization algorithm can be extended as follows:

Step 1. Construct  $m - p$  transfer function models separately between each of the components of  $\mathbf{x}_{2,t}$  and all components of  $\mathbf{x}_{1,t}$  as their input time series. In matrix form, we can formulate this step as

$$\mathbf{x}_{2,t} = \boldsymbol{\nu}_{\mathbf{x}_1\mathbf{x}_2}(B)\mathbf{x}_{1,t} + \boldsymbol{\eta}_{1,t}, \quad (30)$$

where  $\boldsymbol{\nu}_{\mathbf{x}_1\mathbf{x}_2}(B)$  is an  $(m - p) \times p$  matrix and  $\boldsymbol{\eta}_{1,t}$  is vector of residual time series. The frequency transformations related to the model (30) are

$$\mathbf{F}_{\mathbf{x}_1\mathbf{x}_2}(\lambda) = \mathbf{H}_{\mathbf{x}_1\mathbf{x}_2}(\lambda)\mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}(\lambda), \quad (31)$$

$$\mathbf{F}_{\boldsymbol{\eta}_1\boldsymbol{\eta}_1}(\lambda) = \mathbf{F}_{\mathbf{x}_2\mathbf{x}_2}(\lambda) - \mathbf{F}_{\mathbf{x}_2\mathbf{x}_1}(\lambda)\mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}^{-1}(\lambda)\mathbf{F}_{\mathbf{x}_1\mathbf{x}_2}(\lambda), \quad (32)$$

(Priestley, 1982). By replacing  $\mathbf{x}_{2,t}$  in (24) by  $\boldsymbol{\eta}_{1,t}$  we obtain a transfer function model with two orthogonal groups of input time series. Thus, it immediately follows that

$$\mathbf{f}_{\mathbf{x}_1\mathbf{y}}(\lambda) = \mathbf{F}_{\mathbf{x}_1\mathbf{x}_1}(\lambda)\mathbf{h}_{\mathbf{x}_1\mathbf{y}}(\lambda), \quad (33)$$



$$\mathbf{f}_{\eta_1 y}(\lambda) = \mathbf{F}_{\eta_1 \eta_1}(\lambda) \mathbf{h}_{\eta_1 y}(\lambda). \tag{34}$$

The relations (33) and (34) deduce the second step.

step 2. Construct the following transfer function models separately

$$y_t = \boldsymbol{\nu}'_{x_1 y}(\lambda) \mathbf{x}_{1,t} + \eta_{2,t}, \tag{35}$$

$$y_t = \boldsymbol{\nu}'_{\eta_1 y}(\lambda) \boldsymbol{\eta}_{1,t} + \eta_{3,t}. \tag{36}$$

Also, the cross-spectral density vector  $\mathbf{f}_{\eta_1 y}(\lambda)$  is given by

$$\mathbf{f}_{\eta_1 y}(\lambda) = \mathbf{f}_{x_2 y}(\lambda) - \mathbf{F}_{x_2 x_1}(\lambda) \mathbf{F}_{x_1 x_1}^{-1}(\lambda) \mathbf{f}_{x_1 y}(\lambda). \tag{37}$$

Hence, by replacing (32) and (37) into (34), we can obtain

$$\begin{aligned} \mathbf{h}_{\eta_1 y}(\lambda) &= \mathbf{F}_{\eta_1 \eta_1}^{-1}(\lambda) \mathbf{f}_{\eta_1 y}(\lambda) \\ &= \mathbf{h}_2(\lambda), \end{aligned} \tag{38}$$

where  $\mathbf{h}_2(\lambda)$  is given by (26). Moreover, by using (31), (33) and (38),  $\mathbf{h}_1(\lambda)$  in (27) can be represented as

$$\mathbf{h}_1(\lambda) = \mathbf{h}_{x_1 y}(\lambda) - \mathbf{H}_{x_1 x_2}(\lambda) \mathbf{h}_{\eta_1 y}(\lambda). \tag{39}$$

The equations (38) and (39) lead to the last step.

step 3. Derive  $\boldsymbol{\nu}_1(B)$  and  $\boldsymbol{\nu}_2(B)$  in (24) from the following equations

$$\boldsymbol{\nu}_1(B) = (\boldsymbol{\nu}_{x_1 y} - (\boldsymbol{\nu}_{x_1 x_2} * \boldsymbol{\nu}_{\eta_1 y}))(B),$$

$$\boldsymbol{\nu}_2(B) = \boldsymbol{\nu}_{\eta_1 y}(B),$$

where  $\boldsymbol{\nu}_{x_1 x_2} * \boldsymbol{\nu}_{\eta_1 y}$  is  $p \times 1$  vector with the components  $\sum_{j=1}^{m-p} (\nu_{x_1 i x_2 j} * \nu_{\eta_1 j y})$ ,  $i = 1, 2, \dots, p$ .

### 3 Simulation

In this section, we conduct a simulation study to build a transfer function model using orthogonalization approach. To achieve this purpose, we first explain how to fit a single input transfer function model. We follow the instruction given in Bowerman and O'Connell (1993) to fit a parsimonious form of a single input transfer function model given by (2). Consequently, the corresponding linear representation in terms of impulse response weights can be constructed using the linear equations (3).

The parsimonious form of a transfer function is completely characterized by a proper set of  $(r, s, b)$ . These parameters can be estimated based on the sample cross-correlation between input and output time series. The last step to fit a transfer function model is finding an adequate SARIMA model describing the noise series  $\eta_t$ . Two statistics are used to verify whether a single input transfer function model has been adequately constructed. These statistics are given by

$$Q_1 = N(N+2) \sum_{h=1}^K (N-h)^{-1} \hat{\rho}_{x\eta}^2(h), \quad (40)$$

$$Q_2 = N(N+2) \sum_{h=1}^K (N-h)^{-1} \hat{\rho}_{\eta\eta}^2(h), \quad (41)$$

where  $\hat{\rho}_{x\eta}(h)$  is sample cross-correlation function between an input time series  $x_t$  and residual time series  $\eta_t$ , and  $\hat{\rho}_{\eta\eta}(h)$  is the sample autocorrelation function of the residual time series. These statistics both have chi-square distribution. Small values of  $Q_1$  and  $Q_2$  or equivalently high values of their corresponding p-values, say  $p_1$  and  $p_2$ , indicate that the model has been adequately constructed. The aforementioned model fitting can be performed step by step in SAS. It is worth mentioning that the statistics (40) and (41) may propose more than one adequate models for a given data set. We will illustrate that under different adequate models, the ultimate estimates of impulse response weights are approximately identical.

In order to present numerical results for orthogonalization approach, we simulate data from the following transfer function model

$$y_t = (4 + 4B + B^2)x_{1,t} + \frac{B}{1 + 0.6B}x_{2,t} + \eta_t, \quad (42)$$

where

$$\begin{aligned} x_{1,t} &= (1 - 0.4B)a_{1,t}, \\ x_{2,t} &= (1 - 0.3B)x_{1,t} + \eta_{1,t}, \\ \eta_{1,t} &= (1 - 0.5B)a_{2,t}. \end{aligned}$$

The time series  $\eta_t$ ,  $a_{1,t}$  and  $a_{2,t}$  are assumed to be white noise processes. We follow the steps of orthogonalization approach given in section 2.1.

Step 1. Since the parsimonious form of the transfer function between two input time series  $x_{1,t}$  and  $x_{2,t}$  is predetermined in (42), we just need to obtain the corresponding linear representation in terms of impulse response weights.

Step 2. To perform this step, we need to construct two single input transfer function models separately. Using the statistics (40) for a set of  $N = 2000$  simulated sequences from the model (42), we consider two different adequate cases, namely case 1 and 2. These cases are given as bellow:

Case 1.

$$y_t = \frac{3.924 + 7.620B + 3.326B^2 + 0.437B^3}{1 + 0.668B - 0.020B^2}x_{1,t} + \eta_{2,t},$$

where  $p_1 = 0.2705$ . Also

$$y_t = \frac{0.722 - 0.957B}{1 - 0.094B - 0.458B^2}B\eta_{1,t} + \eta_{3,t},$$

where  $p'_1 = 0.3181$ .

Case 2.

$$y_t = \frac{3.923 + 7.346B + 2.631B^2}{1 + 0.597B - 0.108B^2}x_{1,t} + \eta_{2,t},$$

where  $p_1 = 0.3338$ . Also

$$y_t = \frac{-0.113 + 0.645B - 0.310B^2}{1 + 0.598B}\eta_{1,t} + \eta_{3,t},$$

where,  $p'_1 = 0.3039$ .

To complete this step, we need to transform all constructed models to their corresponding linear representation in terms of impulse response weights.

Step 3. Using the impulse response weights produced in steps 1 and 2, we can compute the impulse response weights of the transfer function model (42). Table 1 describes the last step of orthogonalization approach in two mentioned cases. This table shows the impulse response weights at initial lags relating to case 1 and case 2 separately. To compare to the true values of impulse response weights, we define the following error of estimate operators

$$err1(B) = \nu_{\eta_1 y}(B) - \nu_2(B),$$

$$err2(B) = (\nu_{x_1 y} - \nu_{\eta_1 y} * \nu_{x_1 x_2})(B) - \nu_1(B).$$

Table 2 shows the errors of estimate at initial lags. This table demonstrates that the impulse response weights of model (40) are adequately estimated by orthogonalization algorithm in different cases. Note that the more adequate models through the orthogonalization steps or equivalently the higher p-values imply the less error of estimates.

		Lag									
		0	1	2	3	4	5	6	7	8	9
Case 1	$\nu_{x_1y}$	3.924	4.998	0.069	0.495	-0.329	0.230	-0.160	0.112	-0.078	0.054
	$\nu_{\eta_1y}$	0.00	0.722	-0.889	0.247	-0.384	0.077	-0.169	0.019	-0.075	0.001
	$\nu_{x_1x_2}$	1.000	-0.300	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$\nu_{x_1y} - (\nu_{\eta_1y} * \nu_{x_1x_2})$	4.140	4.010	1.032	0.132	0.078	0.102	0.014	0.070	-0.001	0.042
Case 2	$\nu_{x_1y}$	3.923	5.002	0.067	0.501	-0.292	0.228	-0.168	0.125	-0.093	0.069
	$\nu_{\eta_1y}$	-0.113	0.712	-0.736	0.440	-0.263	0.157	-0.094	0.056	-0.033	0.020
	$\nu_{x_1x_2}$	1.000	-0.300	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	$\nu_{x_1y} - (\nu_{\eta_1y} * \nu_{x_1x_2})$	4.250	4.068	0.936	-0.018	0.019	0.042	-0.056	0.058	-0.053	0.045

Table 1: Initial impulse response weights of model (40) estimated by the orthogonalization algorithm.

		Lag									
		0	1	2	3	4	5	6	7	8	9
Case 1	$err1$	0.000	-0.278	-0.289	-0.113	-0.168	-0.052	-0.092	-0.027	-0.048	-0.015
	$err2$	0.140	0.010	0.032	0.132	0.078	0.102	0.014	0.070	-0.001	0.042
Case 2	$err1$	0.250	0.068	-0.064	-0.018	0.019	0.042	-0.056	0.058	-0.053	0.045
	$err2$	0.000	-0.288	-0.136	-0.316	-0.047	0.028	-0.017	0.010	-0.006	0.004

Table 2: The error of estimates of impulse response weights in model (40).

## 4 Modelling of Hydrological Data

In this section we present a study using hydrological data to illustrate the methodology in this paper.

Hydrologists are interested to model the suspended sediment load of a river in terms of some effective environmental parameters. The information on sediment load is useful for designing reservoirs, dams and stable channels, protection of fish and wildlife habitats and watershed management. In practice, fitting a transfer function model with the sediment load as output time series and, the flow discharge and rainfall as input time series can be very useful for forecasting the amounts of the sediment load. In this section we utilize the orthogonalization approach to fit the mentioned transfer function model. To work with real data, the logarithm of monthly amounts of sediment load, flow discharge and rainfall for 25 years [1986-2012] were used from Kasgan river outlets, a branch of the Karoon river located in south of Isfahan province, Iran.

To follow the 3 steps of orthogonalization approach, we first need to fit a transfer function between flow discharge and rainfall, say  $x_{1,t}$  and  $x_{2,t}$  respectively. It is obvious that " $x_{1,t} \sim C x_{2,t}$ ". Therefore we assume the flow discharge as output variable and rainfall as input variable. The

fitted transfer function is as follows:

$$\begin{aligned} \nabla^{12}x_{1,t} &= \nu_{x_1x_2}(B)\nabla^{12}x_{2,t} + \eta_{1,t} \\ &= \frac{0.066 + 0.0316B + 0.041B^2}{1 + 0.062B - 0.278B^2}\nabla^{12}x_{2,t} + \eta_{1,t}, \end{aligned} \tag{43}$$

where " $\nabla^{12}$ " stands for seasonal difference with period 12. The p-value associated with the statistic (40) is  $p_1 = 0.74$  that shows the adequacy of the transfer function model. Note that we do not need to fit a SARIMA model to the residual time series  $\eta_{1,t}$  at this step. The initial impulse response weights of  $\nu_{x_1x_2}(B)$  are also depicted in table 3.

Lag	0	1	2	3	4	5	6	7	8	9
$\nu_{x_1x_2}$	0.066	-0.027	-0.069	0.004	.0017	0	-0.004	0	0.001	0

Table 3: The initial impulse response weights of  $\nu_{x_1x_2}(B)$ .

The next step is fitting two single input transfer function models separately; the sediment load  $y_t$  as output variable and,  $x_{2,t}$  and the residual time series  $\eta_{1,t}$ , produced in step 1, as input variables. The models are given respectively as follows:

$$\begin{aligned} \nabla^{12}y_t &= \nu_{yx_2}(B)\nabla^{12}x_{2,t} + \eta_{2,t} \\ &= B^{12}\frac{-0.039 - 0.043B - 0.061B^2}{1 + 0.582B + 0.967B^2}\nabla^{12}x_{2,t} + \eta_{2,t} \end{aligned} \tag{44}$$

and

$$\begin{aligned} \nabla^{12}y_t &= \nu_{y\eta_1}(B)\eta_{1,t} + \eta_{3,t} \\ &= B^{12}\frac{-0.279 - 0.566B - 0.382B^2 + 0.091B^3}{1 - 1.956B + 0.961B^2}\nabla^{12}\eta_{1,t} + \eta_{3,t}. \end{aligned} \tag{45}$$

To confirm the adequacy of models (44) and (45) the p-value associated with the statistic (40) is computed; the p-values are  $p_1 = 0.86$  and  $p_1 = 0.56$  respectively. The initial impulse response weights of  $\nu_{yx_2}(B)$  and  $\nu_{y\eta_1}(B)$  are also depicted in table (4).

Lag	0	1	2	3	4	5	6	7	8	9
$\nu_{yx_2}$	-0.039	0.021	0.035	0.040	0.057	0.000	0.000	0.000	0.000	0.000
$\nu_{y\eta_1}$	-0.082	-0.035	0.103	-0.156	0.258	0.000	0.000	0.000	0.000	0.000

Table 4: The initial impulse response weights of  $\nu_{yx_2}(B)$  and  $\nu_{y\eta_1}(B)$ .

Therefore, the target transfer function with two inputs can be constructed as follows

$$\nabla^{12}y_t = \nu_{y\eta_1}(B)\nabla^{12}x_{1,t} + (\nu_{yx_2}(B) - \nu_{y\eta_1} * \nu_{x_1x_2}(B))\nabla^{12}x_{2,t} + \eta_t, \tag{46}$$

where  $\eta_t$  is residual time series.

To finalize the model fitting, we fit a SARIMA model to the residual time series  $\eta_t$ . An adequate model is  $(2, 0, 2) \times (1, 1, 1)_{12}$ . The p-value associated to the statistic (41) for this model is  $p_2 = 0.91$ . Thus, model (46) can be rewritten as

$$\begin{aligned} \nabla^{12}y_t &= \nu_{y\eta_1}(B)\nabla^{12}x_{1,t} + (\nu_{yx_2}(B) - \nu_{y\eta_1} * \nu_{x_1x_2}(B))\nabla^{12}x_{2,t} \\ &+ \frac{(1 - 0.107B^{12})(1 - 0.735B - 0.204B^2)}{(1 - 0.922B^{12})(1 - 0.561B - 0.438B^2)}z_t \end{aligned}$$

## 5 Discussion

In this article, we provided an orthogonalization algorithm to construct a multiple input transfer function model. The underlying theory of this technique obtained in the frequency domain but the algorithm was given directly in the time domain based upon convolution of impulse response weights. Also, the numerical results strongly supported the orthogonalization algorithm. In common methods of constructing a multiple input transfer function model, some useful information about the relation among associated time series is ignored. In order to receive more information with a simple algorithm, we thoroughly recommend utilizing the orthogonalization approach to construct a multiple input transfer function model.

## Acknowledgment

The authors would like to thank the editor and referees for their valuable and constructive comments that led to a considerably improved manuscript.

## References

- Bibi, A. (2006), Evolutionary transfer functions of bilinear processes with time-varying coefficients. *Computers and Mathematics with Applications*, **52**(3-4), 331-338.
- Box, G. E. P. and Jenkins, G. M. (1970), *Time series analysis, Forecasting and Control*. Oakland, CA: Holden-Day Inc.

- Cai, Z., Fan, J., and Yao, Q. (2000), Functional-coefficient regression models for nonlinear time series. *J. Amer. Statist. Assoc.*, **95**, 941-956.
- Edlund, P. (1989), On identification of transfer function models by biased regression methods. *Journal of Statistical Computation and Simulation*, **31**(3), 131-148.
- Fan, J. and Yao, Q. (2003), *Nonlinear time series: Nonparametric and Parametric methods*, New York: Springer.
- Liu, L. M. and Hanssen, D. M. (1982), Identification of multiple input transfer function models. *Communications in Statistics - Theory and Methods*, **11**(3), 297-314.
- Moura, M. S., Morettin, P. A., Tolai, M. C., and Chiann, C. (2012), Transfer function models with time varying coefficients. *Journal of Probability and Statistics*, **2012**, 31 pages.
- Muller, D. and Wei, W. W. S. (1997), Iterative least squares estimation and identification of the transfer function model. *Journal of Time series Analysis*, **18**(6), 579-592.
- Nason, G. P., Von Sachs, R. and Kroisandt, G. (2000), Wavelet processes and adaptive estimation of the evolutionary wavelet spectrum. *Journal of the Royal Statistical Society, Series B*, **62**(2), 271-292.
- Park, J. H. (2011), Dimension reduction transfer function model. *Journal of Statistical Computation and Simulation*, **81**(12), 2131-2140.
- Park, J. H., Sriram, T. N., and Yin, X. (2009), Central mean subspace in time series. *J. Comput. Graph. Stat.*, **18**(3), 717-730.
- pas2 Park, J. H., Sriram, T. N., and Yin, X. (2010), Dimension reduction in time series. *Statist. Sinica*, **20**(2), 747-770.
- Priestley, M. B. (1982), *Spectral analysis and time series*, Vol. 1, Academic press.
- Pukkila, T. (1982), On the Identification of Transfer Function Noise Models with Several Correlated Inputs. *Scandinavian Journal of Statistics*, **9**(3), 139-146.

Wei, W. W. S. (2006), Time series analysis, univariate and multivariate methods, 2nd ed. New York: Pearson Education Inc.

Zho, R. and Chon, K. H. (2004), Robust algorithm for estimation of time-varying transfer functions, IEEE Trans. Biom. Eng., **51**(2), 219-228.