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The Beta-Weibull-Logarithmic Distribution: Some Properties and Applications

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Abstract. In this paper, we introduce a new five-parameter distribution with increasing, decreasing, bathtub-shaped failure rate called the Beta-Weibull-Logarithmic (BWL) distribution. Using the Sterling Polynomials, various properties of the new distribution such as its probability density function, its reliability and failure rate functions, quantiles and moments, Rényi and Shannon entropies, moments of order statistics, Bonferroni and Lorenz curves were derived. then the maximum likelihood estimation of BWL distribution for the parameters of BWL distribution are found. Finally the usefulness of this distribution for real data are presented.

Keywords. Beta distribution, Beta-Weibull-Logarithmic distribution, Hazard rate function, Maximum likelihood estimation, Sterling Polynomials, Weibull distribution

MSC: 62E10.

1 Introduction

Let $G(x, \theta)$ be the cumulative distribution function (cdf) of an absolutely continuous random variable following the distribution G, where $\theta \in \Theta$ is the parameter vector. A general class generated from the logit of a beta random variable has been introduced

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by Eugene (2002) with

$$F(x;a,b,\theta) = I_{G(x,\theta)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x,\theta)} t^{a-1}(1-t)^{b-1}dt, \tag{1.1}$$

for $a > 0, b > 0$ and $\theta \in \Theta$, where $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio and $B_y(a,b) = \int_0^y u^{a-1}(1-u)^{b-1}du$ denotes the incomplete beta function. In fact, if V is a random variable from the beta distribution with parameters a and b, then the cdf of a random variable $X = G_\theta^{-1}(V)$ coincides with the cdf (1.1). A random variable X with the cdf (1.1) is said to have a beta-G(BG) distribution and will be denoted by $X \sim BG(a,b,\theta)$. Some special cases of BG distributions are given below.

(1) If $G(x,\theta)$ is the cdf of a standard uniform distribution, then the cdf given in Equation (1.1) yields the cdf of a beta distribution with parameters a and b.

(2) If a is an integer value and $b = n - a + 1$, then the cdf (1.1) becomes as

$$\begin{aligned} F(x,a,b,\theta) &= \frac{1}{B(a,n-a+1)} \int_0^{G(x,\theta)} t^{a-1}(1-t)^{n-a}dt \\ &= \sum_{k=a}^{n} \binom{n}{k} \big(G(x,\theta)\big)^k \,[1-G(x,\theta)]^{n-k}, \end{aligned}$$

which is really the cdf of the a-th order statistic of a random sample of size n from distribution G.

(3) If $a = b = 1$, then the cdf (1.1) reduces to $F(x,\theta) = G(x,\theta)$.

(4) If $a = 1$, then the cdf (1.1) reduces to $F(x,b,\theta) = 1 - [1 - G(x,\theta)]^b$.

(5) If $b = 1$, then the cdf (1.1) reduces to $F(x,a,\theta) = [G(x,\theta)]^a$.

In addition,

$$f(x;a,b,\theta) = \frac{g(x,\theta)}{B(a,b)} \left(G(x,\theta)\right)^{a-1} \left(1 - G(x,\theta)\right)^{b-1} \tag{1.2}$$

and

$$h(x;a,b,\theta) = \frac{g(x,\theta) \left(G(x,\theta)\right)^{a-1} \left(1 - G(x,\theta)\right)^{b-1}}{B(a,b)(1 - I_G(x,\theta)(a,b))}, \tag{1.3}$$

respectively, where $g(x, \theta)$ is the density function corresponding to cdf $G(x, \theta)$. In this study, we attempt to generalize the Webull-Logarithmc (WL) distribution of Ciumara (2009) by taking $G(x, \theta)$ in Equation (1.1) to be the cdf of a WL distribution that will be given in Equation (2.1). Ciumara (2009) compounded a Weibull distribution with a logarithmic distribution and derived a new lifetime distribution with more flexibility than the Weibull distribution. In this study, we propose a new five-parameter distribution, referred to as the Beta-Weibull-Logarithmic (BWL) distribution.

The major reasons for introducing this distribution are as follows.

1. The quality of procedures utilized in statistical analysis heavily depends on the assumed probability model or distributions. Therefore, considerable effort has gone in the development of large classes of standard probability distributions along with relevant statistical methodologies. In fact, there are many continuous univariate distributions in statistical literature, but their applications have not produced useful results in the environmental, financial, biomedical sciences, engineering and economic areas. Therefore, the extension of existing distribution is essential for applications.

2. The WL distribution does not provide a reasonable parametric fit for modeling phenomenon with decreasing, non-linear increasing, or non-monotone failure rates such as the bathtub shape, which are common in firm ware reliability modeling and biological studies.

3. The BWL distribution has greater tail flexibility than the WL distribution. The most realistic hazard rate is bathtub-shaped. This occurs in most real-life systems. Such hazard rates can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

4. The new proposed five-parameter distribution contains many flexible lifetime distributions as special sub-models. These models include the Weibull-Logarithmic (WL), Exponentiated Weibull (EW), Generalized Exponential (GE), Beta-Weibull (BW) and Beta-Exponential (BE) distributions.

The paper is organized as follows. In Section 2, the BWL distribution is defined. The density, survival and hazard rate functions and some of their properties are also given in this section. In Section 3, moments of the BWL distribution were derived. Rényi and Shannon entropies of the BWL distribution are given in Section 4. The moments of order statistics of the BWL distribution are given in Section 5. The Bonferroni and Lorenz curves of the BWL distribution are outlined in Section 6. In Section 7, the model parameters are estimated by the maximum likelihood method, and the asymptotic distribution of estimators are also discussed. In Section 8, the flexibility and potentiality of the new distribution are demonstrated utilizing a real data set and the new model will be compared with some sub-models by various tools. Some concluding remarks are given in Section 9.

2 The BWL distribution

Consider the WL distribution of Ciumara (2009) with the cdf

$$G(x;\gamma,\beta,p) = 1 - \frac{log(1-pe^{-(\beta x)^{y}})}{log(1-p)}, \qquad x>0, \tag{2.1}$$

where $\gamma > 0, \beta > 0$ and $p \in (0,1)$. Substituting $G(x;\theta)$ in Equation (1.1) by the cdf (2.1) yields a new cdf as

$$F(x;a,b,\gamma,\beta,p) = \frac{1}{B(a,b)} \int_0^{G(x;\gamma,\beta,p)} t^{a-1}(1-t)^{b-1}, \tag{2.2}$$

where $a > 0$, $b > 0$,$\gamma > 0$,$\beta > 0$ and $p \in (0,1)$. A random variable X with the cdf (2.2) is said to have a BWL distribution and will be denoted by $X \sim BWL(a,b,\gamma,\beta,p)$. From Equations (1.2) and (1.3), the pdf and failure functions of a BWL distribution are given, respectively, by

$$f(x;a,b,\gamma,\beta,p) \tag{2.3}$$

$$= \frac{p\gamma\beta^{\gamma}x^{\gamma-1}e^{-(\beta x)^{\gamma}}\left(log\left(1-pe^{-(\beta x)^{\gamma}}\right)\right)^{b-1}}{B(a,b)\left(log(1-p)\right)^{a+b-1}} \times \frac{\left(log\left(\frac{1-p}{\left(1-pe^{-(\beta x)^{\gamma}}\right)}\right)\right)^{a-1}}{\left(pe^{-(\beta x)^{\gamma}}-1\right)}, \quad x>0, \tag{2.4}$$

and

$$h(x;a,b,\gamma,\beta,p) = \frac{p\gamma\beta^{\gamma}x^{\gamma-1}e^{-(\beta x)^{\gamma}}\left(log\left(1-pe^{-(\beta x)^{\gamma}}\right)\right)^{b-1}}{B(a,b)\left(log(1-p)\right)^{a+b-1}}$$
$$= \times\frac{\left(log\left(\frac{1-p}{(1-pe^{-(\beta x)^{\gamma}})}\right)\right)^{a-1}}{\left(pe^{-(\beta x)^{\gamma}}-1\right)I_{\frac{log\left(1-pe^{-(\beta x)^{\gamma}}\right)}{log(1-p)}}(b,a)}, \quad x>0$$

The survival function of BWL distribution are given by

$$S(x;a,b,\gamma,\beta,p) = I_{\frac{log\left(1-pe^{-(\beta x)^{\gamma}}\right)}{log(1-p)}}(b,a),$$

where $x > 0$. Using the series representation

$$(1-z)^{b-1} = \sum_{j=0}^{\infty}\frac{\Gamma(b)(-1)^{j}}{\Gamma(b-j)j!}z^{j},$$

the cdf of BWL distribution is given by

$$F_{BWL}(x) = \frac{1}{B(a,b)}\int_{0}^{G(x,\gamma,\beta,p)} t^{a-1}(1-t)^{b-1}dx.$$

For the real non-integer $b > 0$ and $|z| < 1$, the $F_{BWL}(x)$ can be written as

$$F_{BWL}(x) = \frac{1}{B(a,b)}\int_{0}^{G(x,y,\beta,p)} t^{a-1}\sum_{k=0}^{\infty}\binom{b-1}{k}(-t)^{k}dx \tag{2.5}$$
$$= \frac{1}{bB(a,b)}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{(a+k)B(k+1,b-k)}\left(\frac{log\left(\frac{1-p}{1-pe^{-(\beta x)^{\gamma}}}\right)}{log(1-p)}\right)^{a+k}.$$

It is obvious from Equation (2.3) that the BWL distribution is much more flexible than the WL distribution. Figures 1 and 2 plot the density and failure-rate functions of the BWL distribution for some choices of the parameter values, including some well-known lifetime distributions. As can be seen from Figure 2, a characteristic of the BWL distribution is that its failure rate function can be decreasing, increasing, bathtub, or unimodal, depending on its parameter values.

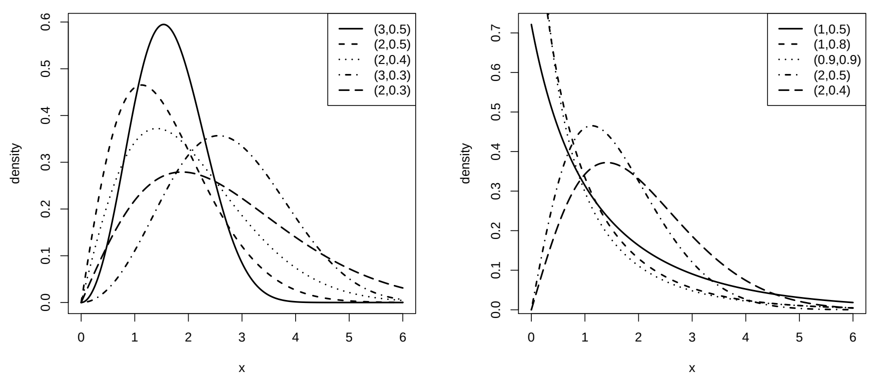

Figure 1: Plots of the density function of BWL distribution for $a = b = 1$, $\theta = 0.5$, and for different values of (γ, β).

2.1 Special distributions

Some special lifetime distributions are achieved from the BWL distribution as follows.

(1) The WL distribution is obtained when $a = b = 1$. If, in addition, $p \to 0^+$,The Weibull distribution with parameters $\gamma > 0$ and $\beta > 0$ is achieved. Clearly, if $\gamma = 1$ in this case, the exponential distribution with parameter $\beta > 0$ is obtained.

(2) If $a = 1$, then we obtain a new lifetime distribution belonging to the frailty parameter family with the cdf

$$F(x; b, \gamma, \beta, p) = 1 - \left(\frac{log\left(1 - pe^{-(\beta x)^{\gamma}}\right)}{log(1-p)} \right)^{b}, \quad x > 0 \tag{2.6}$$

If, in addition, $p \longrightarrow 0^+$, then Equation (2.6) reduces to the cdf of a Weibull distribution with parameters $b^{\frac{1}{\gamma}}\beta > 0$ and $\gamma > 0$.

(3) If $b = 1$, then the BWL distribution gives a new lifetime distribution with the cdf

$$F(x; a, \gamma, \beta, p) = \left(1 - \frac{log\left(1 - pe^{-(\beta x)^{\gamma}}\right)}{log(1-p)} \right)^{a}, \quad x > 0, \tag{2.7}$$

which belongs to the resilience parameter family. In addition, if $p \longrightarrow 0^+$, then Equation (2.7) reduces to an Exponentiated Weibull (EW) distribution with parameters $a > 0, \gamma > 0$ (Mudholkar , 1995) and with the pdf

$$f_{EW}(x; a, \gamma, \beta) = a\gamma\beta^{\gamma} x^{\gamma-1} e^{-(\beta x)^{\gamma}} \left(1 - e^{-(\beta x)^{\gamma}}\right)^{a-1}, \quad x > 0.$$

Clearly, the EW distribution gives the Genaralized Exponential (GE) distribution (Gupta , 1999) when $\gamma = 1$,

$$f_{GE}(x; a, \beta) = a\beta e^{-\beta x}\left(1 - e^{-\beta x}\right)^{a-1}, \quad x > 0.$$

(4) If $p \to 0^+$, then Equation (2.3) reduces to the df of a BW distribution (Famoye , 2005) as

$$f_{BW}(x; a, b, \gamma, \beta) = \frac{\gamma\beta^{\gamma}}{B(a, b)} x^{\gamma-1} e^{-b(\beta x)^{\gamma}}\left(1 - e^{-(\beta x)^{\gamma}}\right)^{a-1}, \quad x > 0. \tag{2.8}$$

Furthermore, if $\gamma = 1$, then Equation (2.8) includes the cdf of a BE distribution (Nadarajah , 2004, 2006) with

$$f_{BE}(x; a, b, \beta) = \frac{\beta}{B(a, b)} e^{-b\beta x}\left(1 - e^{-\beta x}\right)^{a-1}, \quad x > 0.$$

2.2 Simulation and quantiles

Assume that V is a beta random variable with parameters a and b. Utilizing equation $X = G_{\theta}^{-1}(V)$, where $\theta = (\gamma, \beta, p)$, the simulation of a BWL distribution can be easily attained by the relationship $X = -\frac{1}{\beta}\left[log\left(\frac{1-v}{1-pv}\right)\right]^{\frac{1}{\alpha}}$. The q-th quantile of a BWL distribution is obtained by solving the non-linear equation $I_{G(m,\gamma,\beta,p)}(a, b) = q$, where $G(m, \gamma, \beta, p) = \frac{1-e^{-(\beta m)^{\gamma}}}{1-pe^{-(\beta m)^{\gamma}}}$. Particularly, the median is immediately achieved by setting $q = 0.5$ in the above non-linear equation, which can be numerically calculated using MAPLE or MATLAB software.

Proposition 2.1. *The limiting distribution of BWL(a, b, γ, β, p), when $p \longrightarrow 0^+$, is*

$$\lim_{p \to 0^+} F(x) = \int_0^{1-e^{-(\beta x)^{\gamma}}} u^{a-1}(1 - u)^{b-1} du$$

which is the cdf of BW distribution.

Proof. The proof is a forward calculation and is omitted. □

Proposition 2.2. *The limiting behavior of hazard function of BWL distribution is*

(1) *for* $a = b = 1$ *and* $\gamma = 1$, $\lim_{x\to 0} h(x) = \begin{cases} \frac{\beta}{(p-1)\log(1-\theta)}, & 0 < p < 1, \\ -\infty, & p = 1, \\ 0, & p = 1 \end{cases}$ *and* $\lim_{x\to\infty} h(x) = \frac{\beta}{p}$.

(2) *for* $a = b = 1$ *and* $\gamma > 1$, $\lim_{x\to 0} h(x) = 0$, *for each* $\beta > 0, 0 < p < 1$ *and* $\lim_{x\to\infty} h(x) = \infty$.

(3) *for* $a = b = 1$ *and* $0 < \gamma < 1$, $\lim_{x\to 0} h(x) = \infty$, $\lim_{x\to\infty} h(x) = 0$.

Proof. As the proof contains straightforward calculations, it is omitted. □

Theorem 2.1. *For each* $\delta \in R$ *and* $|z| < 1$,

$$[-\log(1-z)]^{\delta} = \sum_{m=0}^{\infty} \rho_m(\delta) z^{\delta+m},$$

where $\rho_0(\delta) = 1$, *and for each* $m \geq 1$, $\rho_m(\delta) = \delta\psi_{m-1}(m+\delta-1)$, *and the coefficients of* $\psi_m(.)$, *Sterling polynomials that satisfies their name, are as*

$$\begin{aligned} \psi_{n-1}(w) \quad = \quad & \frac{(-1)^{n-1}}{(n+1)!}\Big[H_n^{n-1} - \frac{w+2}{n+2}H_n^{n-2} + \frac{(w+2)(w+3)}{(n+2)(n+3)}H_n^{n-3} - \cdots \\ & +(-1)^{n-1}\frac{(w+2)(w+3)\cdots(w+n)}{(n+2)(n+3)\cdots(2n)}H_n^0\Big], \end{aligned}$$

where H_n^m *are positive integers and the following conditions apply.*

$$\begin{aligned} & H_0^0 = H_{n+1}^n = 1, \quad H_{n+1}^m = (2n+m-1)H_n^m + (n-m+1)H_n^{m-1} \\ & H_{n+1}^0 = 1 \times 3 \times 5 \times \cdots \times (2n+1). \end{aligned}$$

Proof. It follows from the results by Ward (1934). □

3 The moments of BWL distribution

Now, with the change of variables $u = 1 - \frac{\log\left(1-pe^{-(\beta x)^{\gamma}}\right)}{\log(1-p)}$, r-th order central moment of the distribution is obtained as follows.

$$EX^r = \frac{(\log p)^{\frac{r}{\gamma}}}{\beta^r B(a,b)} \int_0^1 \left(1 - \frac{\log\left(1-(1-p)^{1-u}\right)}{\log p}\right)^{\frac{r}{\gamma}} u^{a-1}(1-u)^{b-1}du.$$

According to $\left|\frac{\log\left(1-(1-p)^{1-u}\right)}{\log p}\right| < 1$,

$$EX^r = \frac{(\log p)^{\frac{r}{\gamma}}}{\beta^r B(a,b)} \int_0^1 \left(\sum_{k=0}^{\infty} \binom{\frac{r}{k}}{k} (-1)^k \frac{1}{(\log p)^k} \left[\log\left(1-(1-p)^{1-u}\right)\right]^k\right) u^{a-1}(1-u)^{b-1}du.$$

Using theorem 2.1 for $\left[\log\left(1-(1-p)^{1-u}\right)\right]^k$,

$$EX^r = \frac{(\log p)^{\frac{r}{\gamma}}}{\beta^r B(a,b)} \sum_{k=0}^{\infty} \binom{\frac{r}{\gamma}}{k} \sum_{m=0}^{\infty} \rho_m(k) \int_0^1 (1-p)^{(k+m)(1-u)} u^{a-1}(1-u)^{b-1}du$$

Considering $(1-p)^{(k+m)(1-u)} = e^{(k+m)(1-u)\log(1-p)}$ and to help Maclaurin expansion $e^{(k+m)(1-u)\log(1-p)}$,

$$EX^r = \frac{(\log p)^{\frac{r}{\gamma}}}{\beta^r B(a,b)} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \binom{\frac{r}{\gamma}}{k} \frac{\rho_m(k)(k+m)^j[\log(1-p)]^j}{j!(\log p)^k} B(a, b+j). \tag{3.1}$$

4 Moments for Order statistics

We now derive an explicit expression for the density function of the i-th order statistic $X_{i:n}$ in a random sample of size n from the BWL distribution. According to Bidram (2013),

$$f_{i:n}(x) = \frac{g(x)\left(G(x)\right)^{a-1}\left(1-G(x)\right)^{b-1}}{B(a,b)B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \sum_{m=0}^{\infty} c_{i+j-1,m}\left(G(x)\right)^m, \tag{4.1}$$

where the coefficients $c_{i+j-1,m}$ follow from Equation $c_{j,i} = (ia_0)^{-1} \sum_{m=1}^{i} (jm - i + m) a_m c_{j,i-m}$ and $c_{j,0} = a_0^j$. Hence, $c_{j,i}$ can be calculated from $c_{j,1}, \cdots, c_{j,i-1}$ and then from $a_0, \cdots, a_i$, where $a_m = \frac{(1-b)_m}{B(a,b)(a+m)m!}$ and $(f)_k = f(f+1)\cdots(f+k-1)$.

Combining Equations (2.3) and (4.1), the pdf of the i-th order statistic of the BWL distribution can be easily obtained by

$$f_{i:n}(x) = \sum_{m=0}^{\infty} K_{i,n,m} f_{BWL}(x, a, b+r, \gamma, \beta, p), \tag{4.2}$$

where

$$K_{i,n,m} = \frac{\Gamma(n-i+1)\Gamma(m+1)(-1)^{i+r}}{\Gamma(n-i-j)\Gamma(m-r+1)i!j!B(i,n-i+1)} c_{i+j-1,m}.$$

Equation (4.2) shows that the density function of the BWL order statistics can be expressed as a linear combination of the pdf of the BWL. Moments of order statistics play an important role in quality control testing and reliability, where a practitioner needs to predict the failure of future items based on the times of a few early failures. These predictors are often based on moments of order statistics. The ordinary, inverse and factorial moments of the BWL order statistics can be calculated from a weighted infinite linear combination of those quantities for BWL distributions. For example, using Equation (3.1), we immediately obtain the r-th generalized moment of $X_{i:n}$ as

$$E(X_{i:n}^r) = \frac{(\log p)^{\frac{r}{\gamma}}}{\beta^r} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{[b+r]_r \rho_m(s) \Gamma\left(\frac{r}{\gamma}+1\right)(1-p)^{k+s}}{[a+b+r]_r \Gamma\left(\frac{r}{\gamma}-k+1\right)(\log p)^k k!} K_{i,n,m},$$

where $[f]_r = f(f-1)\cdots(f-r)$.

5 The Bonferroni and Lorenz curves

For a random variable X with cumulative distribution function $F(.)$ and probability density function, the Bonferroni curve is defined as

$$B_F(F(x)) = \frac{1}{\mu F(x)} \int_0^x u f(u) du.$$

Therefore, for the BWL,

$$B_F(F(x) = \frac{1}{\mu F(x)}\int_0^x uf(u)du = \frac{(\log p)^{\frac{1}{\gamma}}}{\mu\beta B(a,b) I_{\frac{\log\left(1-e^{-(\beta x)^\gamma}\right)}{\log(1-p)}}(a,b)}$$
$$\times \sum_{i=0}^{b-1}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\Big[\binom{b-1}{i}\binom{\frac{1}{\gamma}}{j}\left(\frac{1}{(m+j)\log(1-p)}\right)^{a+i}\left(\frac{1}{\log p}\right)^{j}(-1)^{2i+2j+a-1}$$
$$\times \rho_m(j)\left(\gamma\left(a+i,(m+j)\log(1-p)\right)-\gamma\left(a+i,(m+j)\log\left(1-pe^{-(\beta x)^\gamma}\right)\right)\right)\Big].$$

The Lorenz curve of the BWL distribution can be obtained through the expression

$$L_F(F(x) = B_F[F(x)]F(x) = \frac{1}{\mu}\int_0^x uf(u)du$$
$$= \frac{(\log p)^{\frac{1}{\gamma}}}{\mu\beta B(a,b)}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\Big[\binom{b-1}{i}\binom{\frac{1}{\gamma}}{j}\left(\frac{1}{(m+j)\log(1-p)}\right)^{a+i}\left(\frac{1}{\log p}\right)^{j}(-1)^{2i+2j+a-1}$$
$$\times \rho_m(j)\left(\gamma\left(a+i,(m+j)\log(1-p)\right)\gamma\left(a+i,(m+j)\log\left(1-pe^{-(\beta x)^\gamma}\right)\right)\right)\Big],$$

where μ is the mean of the BWL distribution and $\gamma(r,s) = \int_0^s x^{r-1}e^{-x}dx$ is the incomplete gamma function.

5.1 Estimation and inference

In this section, the estimation of the parameters of the BWL distribution will be discussed. Let $X_1, X_2, \ldots, X_n$ be a random sample with observed values $x_1, x_2, \ldots, x_n$ from BWL distribution with parameters a, b, γ, β and p. Let $\Theta = (a, b, \gamma, \beta, p)^T$ be the parameter vector. The total log-likelihood function is given by

$$\begin{aligned}
l_n &\equiv l_n(x;\Theta) = n\log\gamma + n\gamma\log\beta + (\gamma-1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}(\beta x)^\gamma \\
&+ (a-1)\sum_{i=1}^{n}\log\left[\log\left(\frac{1-p}{1-pe^{-(\beta x)^\gamma}}\right)\right] - (b-1)\sum_{i=1}^{n}\log\left[\log\left(1-pe^{-(\beta x)^\gamma}\right)\right] \\
&- \sum_{i=1}^{n}\log\left(1-pe^{-(\beta x)^\gamma}\right) - (a+b-1)\sum_{i=1}^{n}\log\left(\log(1-p)\right) \\
&+ n\log\Gamma(a+b) - n\log\Gamma(a) - n\log\Gamma(b).
\end{aligned}$$

The associated score function is given by $U_n(\Theta) = \left(\frac{\partial l_n}{\partial a}, \frac{\partial l_n}{\partial b}, \frac{\partial l_n}{\partial \gamma}, \frac{\partial l_n}{\partial \beta}, \frac{\partial l_n}{\partial p}\right)^T$, where

$$\begin{aligned}
\frac{\partial l_n}{\partial \gamma} &= \frac{n}{\gamma} + n \log \beta + \sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} (\beta x_i)^\gamma \log(\beta x_i) \\
&- (a-1) \sum_{i=1}^{n} \frac{p(\beta x_i)^\gamma \log(\beta x_i) e^{(\beta x_i)^\gamma}}{\left(1 - pe^{(\beta x_i)^\gamma}\right)(1-p) \log\left(\frac{1-p}{1-pe^{(\beta x_i)^\gamma}}\right)} \\
&- (b-1) \sum_{i=1}^{n} \frac{p(\beta x_i)^\gamma \log(\beta x_i) e^{(\beta x_i)^\gamma}}{\log\left(1 - pe^{(\beta x_i)^\gamma}\right)} + \sum_{i=1}^{n} \frac{p(\beta x_i)^\gamma \log(\beta x_i) e^{(\beta x_i)^\gamma}}{pe^{(\beta x_i)^\gamma} - 1}.
\end{aligned}$$

$$\frac{\partial l_n}{\partial a} = \sum_{i=1}^{n} \log\left[\log\left(\frac{1-p}{1-pe^{(\beta x_i)^\gamma}}\right)\right] - n \log\left(\log(1-p)\right) + n\Psi(a+b) - n\Psi(a).$$

$$\frac{\partial l_n}{\partial b} = \sum_{i=1}^{n} \log\left[\log\left(1 - pe^{-(\beta x)^\gamma}\right)\right] - n \log\left(\log(1-p)\right) - n\Psi(a+b) - n\Psi(b).$$

$$\begin{aligned}
\frac{\partial l_n}{\partial \beta} &= \frac{n\gamma}{\beta} + \gamma \sum_{i=1}^{n} x_i (\beta x_i)^{\gamma-1} - (b-1) \sum_{i=1}^{n} \frac{p(\beta x_i)^\gamma \log(\beta x_i) e^{(\beta x_i)^\gamma}}{(1 - pe^{(\beta x_i)^\gamma}) \log\left[(1 - pe^{(\beta x_i)^\gamma})\right]} \\
&- (a-1) \sum_{i=1}^{n} \frac{p(\beta x_i)^\gamma \log(\beta x_i) e^{(\beta x_i)^\gamma}}{(1 - pe^{(\beta x_i)^\gamma}) \log\left[(1 - pe^{(\beta x_i)^\gamma})\right] (1-p) \log\left[\frac{1-p}{(1-pe^{(\beta x_i)^\gamma})}\right]} \\
&+ \sum_{i=1}^{n} \frac{p(\beta x_i)^\gamma \log(\beta x_i) e^{(\beta x_i)^\gamma}}{pe^{(\beta x_i)^\gamma} - 1}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial l_n}{\partial p} &= (a-1) \sum_{i=1}^{n} \frac{(1+2p) e^{(\beta x_i)^\gamma} - 1}{(1 - pe^{(\beta x_i)^\gamma}) \log\left[(1 - pe^{(\beta x_i)^\gamma})\right] (1-p) \log\left[\frac{1-p}{(1-pe^{(\beta x_i)^\gamma})}\right]} \\
&- (b-1) \sum_{i=1}^{n} \frac{e^{(\beta x_i)^\gamma}}{(1 - pe^{(\beta x_i)^\gamma}) \log\left(1 - pe^{(\beta x_i)^\gamma}\right)} + \frac{n(a+b-1)}{(1-p)\log(1-p)}.
\end{aligned}$$

$\frac{\partial \log \Gamma(x)}{\partial x} = \Psi(x)$ which is called digamma function.

The MLE of Θ, say $\hat{\Theta}$, is obtained by solving the non-linear system $U_n(\Theta) = 0$. The solution of this non-linear system of equation does not have a closed form. For interval estimations and hypothesis testing on the model parameters, we need the information matrix. The 5×5 observed information matrix is

$$\mathbf{I_n}(\boldsymbol{\Theta}) = -\begin{bmatrix} I_{aa} & I_{ab} & I_{a\gamma} & I_{a\beta} & I_{ap} \\ I_{ba} & I_{bb} & I_{b\gamma} & I_{b\beta} & I_{bp} \\ I_{\gamma a} & I_{\gamma b} & I_{\gamma\gamma} & I_{\gamma\beta} & I_{\gamma p} \\ I_{\beta a} & I_{\beta b} & I_{\beta\gamma} & I_{\beta\beta} & I_{\beta p} \\ I_{pa} & I_{pb} & I_{p\gamma} & I_{p\beta} & I_{pp} \end{bmatrix},$$

where $I_{aa} = \frac{\partial^2 l_n}{\partial^2 a}, I_{ab} = \frac{\partial^2 l_n}{\partial a \partial b}, \cdots, I_{pp} = \frac{\partial^2 l_n}{\partial^2 p}$.

Applying the usual large sample approximation, MLE of Θ, i.e. $\hat{\Theta}$, can be treated as being approximately $N_5\Big(\Theta, \big(J_n(\Theta)\big)^{-1}\Big)$, where $J_n(\Theta) = E\big(I_n(\Theta)\big)$. Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ is $N_5\Big(0, \big(J(\Theta)\big)^{-1}\Big)$ where $J(\Theta) = \lim_{n\to\infty} n^{-1} I_n(\Theta)$ is the unit information matrix. This asymptotic behavior remains valid if $J(\Theta)$ is replaced by the average sample information matrix evaluated at $\hat{\Theta}$, say $n^{-1} I_n(\Theta)$. The estimated asymptotic multivariate normal distribution $N_5\Big(\Theta, \big(J_n(\hat{\Theta})\big)^{-1}\Big)$ of $\hat{\Theta}$ can be used to construct approximate confidence intervals for the parameters and for the hazard rate and survival functions. A $100(1-\gamma)\%$ asymptotic confidence interval for each parameter Θ_r is given by

$$ACI_r = \Big(\hat{\Theta}_r - Z_{\frac{\gamma}{2}} \sqrt{\hat{I_{rr}}}, \hat{\Theta}_r + Z_{\frac{\gamma}{2}} \sqrt{\hat{I_{rr}}}\Big),$$

where $\hat{I_{rr}}$ is the (r, r) diagonal element of $\big(I_n(\hat{\Theta})\big)^{-1}$ for $r = 1, 2, 3, 4, 5$, and $Z_{\frac{\gamma}{2}}$ is the quantile $1 - \frac{\gamma}{2}$ of the standard normal distribution.

6 Applications of the BWL distribution

An application of the BWL distribution using a real data set is presented in this section. The data set ($n = 63$) is on the strengths of 1.5 cm glass fibers and it is obtained from Smit (1987). Barreto-Souza (2010) applied the beta generalized exponential (BGE) distribution to fit the data. The data are: 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

The TTT plot curve for this data in Figure 3 demonstrates an increasing hazard rate function that shows the appropriateness of the BWL distribution for this data set. Then, we fit the Beta Weibull Logarithmic (BWL) distribution defined in Equation (2.3). Its fitness is also compared with WL, BE, BW, GE and EW distributions. Table 1 depicts the MLE's of the unknown parameters for these distributions, the values of the statistics AIC (Akaike Information Criterion), the statistics AICc (Akaike Information Citerion with correction), W (Watson statistic) and CM (Cramérvon Mises statistic) for this data. These values reveal that the BWL distribution provides a better fit than WL, BE, BW, GE and EW distributions for this data set. W and CM statistics were applied in order to see which distribution fits better to this data. The W and CM test statistics are described in details in Chen (1995). Generally, the smaller the values of W and CM, the better the fit to the data. According to these statistics in Table 1, the BWL distribution fit to this data set better than the others. Density plots are illustrated in Figure 4. It is clear that the BWL model provides a better fit than the other models.

Table 1: MLEs, AIC, AICc, CM, W, K-S statistics and p-values for the strengths of 1.5 cm glass fibers.

Distribution	BWL	WL	BE	GE	BW	EW
$\hat{\gamma}$	0.0384	6.158	-	-	7.748	0.5822
$\hat{\beta}$	0.0007	0.5934	22.75	2.613	39.73	7.273
$\hat{p}$	0.9335	0.5	-	-	-	-
$\hat{a}$	1	-	17.43	31.36	0.6199	0.6712
$\hat{b}$	2	-	43.15	-	43.15	-
AIC	33.2	36.2	54	70.8	37.2	35.2
AICc	34.3	36.4	54.4	71	37.9	35.6
W	14.48	15.48	15.76	16.01	15.43	15.41
CM	0.0143	0.2441	0.5677	0.7921	0.1946	0.1920

7 Conclusion

A new five-parameter distribution called the BWL distribution was introduced. This distribution is a generalization of the BW distribution. A characteristic of the BWL distribution is that its failure rate function can be decreasing, increasing, bathtub-shaped and unimodal depending on its parameter values. Various properties of the new distribution such as its probability density function, its reliability and failure rate functions, Rényi and Shannon entropies and moments were obtained. The maximum likelihood estimation procedure was presented. Fitting the BWL model to a real data set demonstrates the flexibility and capacity of the proposed distribution in data modeling. In view of the density and failure rate function shapes, it seems that the proposed model can be considered as a suitable candidate model in reliability analysis, biological systems, data modeling, and related fields.

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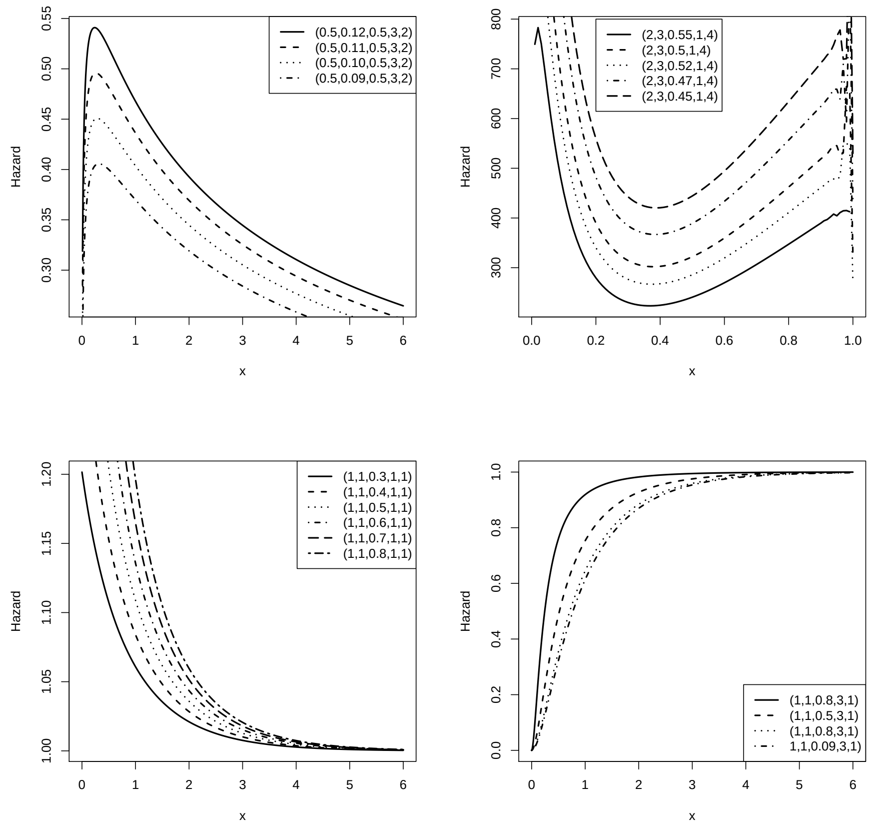

Figure 2: Plots of hazard function of BWL distribution for different values of (γ, β, p, a, b).

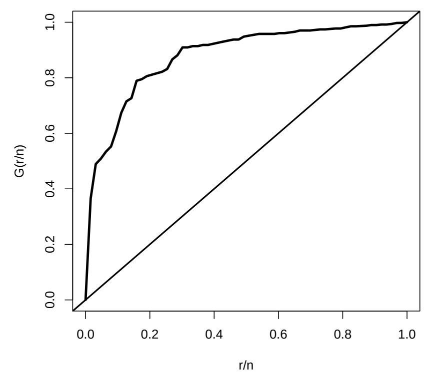

Figure 3: TTT plot curve of strengths of 1.5 cm glass fibers

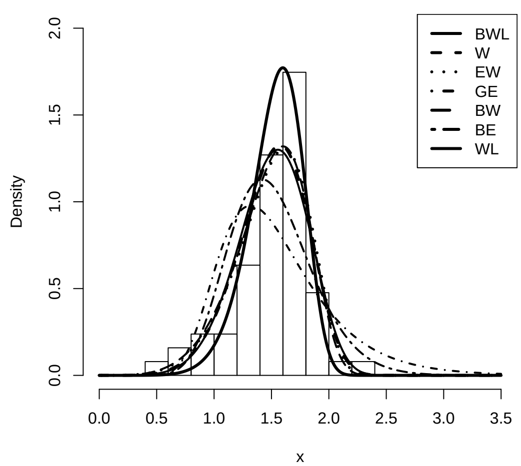

Figure 4: Fitted pdf of the BWL, WL, BE, GE, BW and EW distributions for the data set corresponding to Table 1