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# The Beta Exponentiated Gumbel Distribution

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**Abstract.** We introduce a new five-parameter distribution called the beta exponentiated Gumbel (BEG) distribution that includes the beta Gumbel, exponentiated Gumbel and Gumbel distribution. Expressions for the distribution function, density function and rth moment of the new distribution and order statistics are obtained. We discuss estimation of the parameters by maximum liklelihood and provide the information matrix. Using a real data set, we observe that the BEG distribution is flexible and can be used quite effectively in analysing positive data in place of the special cases.

**Keywords.** Beta Gumbel distribution, Exponentiated Gumbel, Gumbel distribution, Information matrix, Maximum likelihood estimation.

**MSC:** 62E99, 62N99.

## 1 Introduction

Nadarajah and Kotz (2006) defined the cumulative distributions function (cdf) of the exponentiated Gumbel (EF) distribution function by

$$G(x) = 1 - \left\{1 - \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}, \quad x > 0 \tag{1.1}$$

for $-\infty < \mu < +\infty$, $\sigma > 0$ and $\alpha > 0$. Clearly, the Gumbel ($G$) distribution is a particular case of the EG distributions when $\alpha = 1$.

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Eugene et al. (2002) defined a class of generalized distribution from it given by

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1} dw, \quad x > 0 \tag{1.2}$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weight and $B(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}dw$, is the beta function. The cdf $G(x)$ could be quite arbitrary and $F$ is named the beta $G$ distribution. Indeed, if $V$ is a beta distribution with parameter $a$ and $b$, then the cdf of the $X = G^{-1}(V)$ agrees with the cdf given in (1.2).

Some new distributions have been introduced using (1.2) in the literature. For example, the beta normal (*BN*) distribution was introduced by Eugene et al. (2002) with $G(x)$ in (1.2) to be the cdf of a normal distribution. General expressions for the moments of the *BN* distribution were derived by Gupta and Nadarajah (2004). The beta gumbel (*BG*) distribution was introduced by Nadarajah and Kotz (2004) in which $G(x)$ in (1.2) is the cdf of a Gumbel distribution. The Beta Fréchet (*BF*) distribution was introduced by Nadarajah and Gupta (2004) in which $G(x)$ in (1.2) is the cdf of a Fréchet distribution and investigated by Barreto-Souza et al. (2008). Some other examples are the beta exponential (*BE*) distribution introduced by Nadarajah and Kotz (2006), the beta generalized exponential (*BGE*) distribution introduced by Barreto-Souza et al. (2010), the beta Weibul (*BW*) distributeon introduced by Famoye et al. (2005), the beta exponential-geometric (BEG) distribution introduced by Bidram et al. (2013).

In this paper we introduced the beta exponentiated Gumbel (*BEG*) distribution by taking $G(x)$ in (1.2) to be as defined in Equation (1.1). The cdf of the *BEG* distribution is then

$$F(x) = \frac{1}{B(a,b)} \int_0^{1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}} w^{a-1}(1-w)^{b-1} dw, \quad x > 0 \tag{1.3}$$

for $a > 0$ , $b > 0$ ,$-\infty < \mu < +\infty$, $\sigma > 0$ and $\alpha > 0$. $\mu$ is location parameter, $\sigma$ is scale parameter and $\alpha$ is shape parameter. The pdf and the hazard rate function of the new distribution are, respectively,

$$\begin{aligned} f(x) &= \frac{\alpha}{\sigma}\frac{\exp\left(-\frac{x-\mu}{\sigma}\right)}{B(a,b)}\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right] \\ &\times \left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha b-1} \\ &\times \left\{1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}\right\}^{(a-1)}, x>0 \end{aligned} \quad (1.4)$$

and

$$\begin{aligned} h(x) &= \frac{\alpha\exp\left(-\frac{x-\mu}{\sigma}\right)\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha+b-2}}{\sigma B(a,b)} \\ &\times \frac{\left\{1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}\right\}^{(a-1)}}{\left(1-I_{\left(1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}\right)}(a,b)\right)}, \quad x>0. \end{aligned} \quad (1.5)$$

To prove that (1.4) is indeed a probability density function, we need to show that

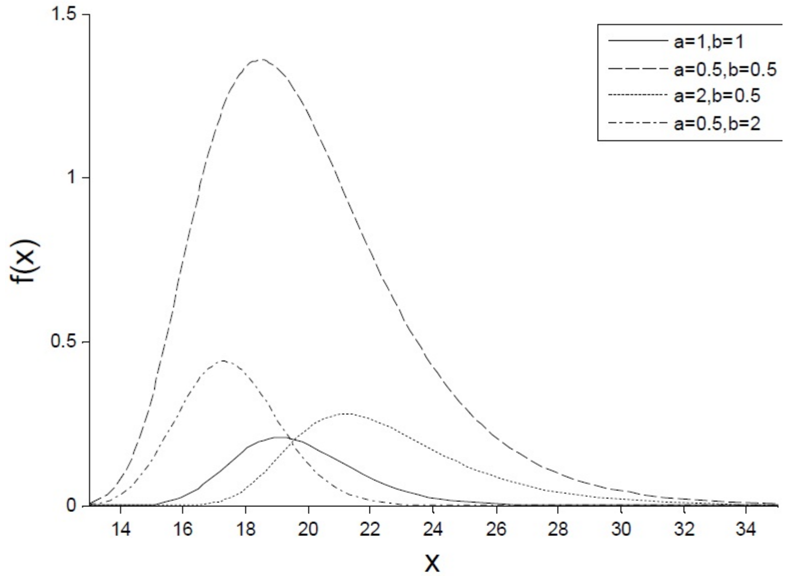
$$\int_{-\infty}^{+\infty}\frac{\alpha}{\sigma}\frac{-\nu}{B(a,b)}\exp(-\nu)\{1-\exp(-\nu)\}^{\alpha b-1}\{1-\{1-\exp(-\nu)\}^{\alpha}\}^{a-1}dx = 1,$$

where $\nu = \exp\left(-\frac{x-\mu}{\sigma}\right)$.

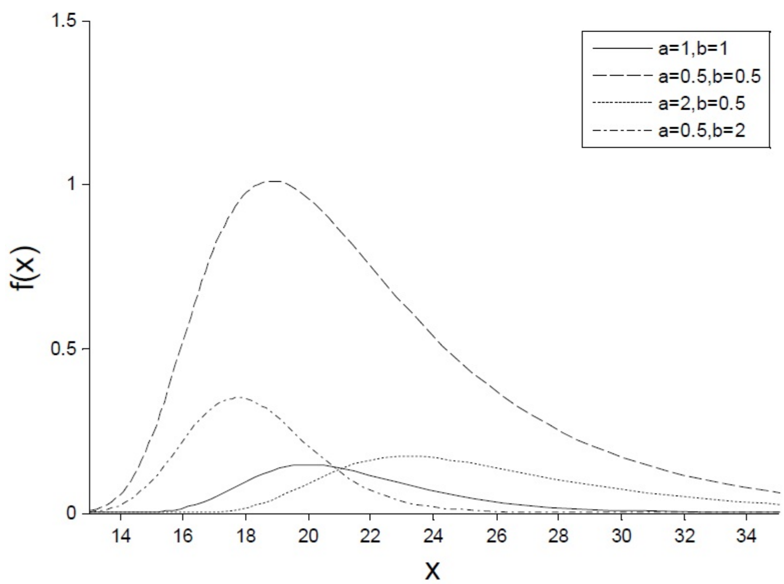
Changing to the variable $u = 1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}$ we obtain

$$\frac{1}{B(a,b)}\int_0^1 u^{a-1}(1-u)^{b-1}du = 1.$$

If $X$ is a random variable with pdf (1.4), we write $X \sim BEG(a,b,\mu,\sigma,\alpha)$. The *BEG* distribution generalizes some well-known distributions in the literature. The *EG* distribution is a special case for the choice $a=b=1$. If in addition $\alpha=1$, we obtain Gumbel distribution. The *BG* distribution obtained from (1.4) with $\alpha=1$. Plots of the density (1.4) for some special value of $a, b, \mu, \sigma$ and $\alpha$ are given in Figure 1.




$(a)\ \mu = 20, \sigma = 2.5, \alpha = 1$




$(b)\ \mu = 20, \sigma = 2.5, \alpha = 2$

Figure 1: Plots of the density (4) for some values of the parameters.

## 2 Distribution Function and Order Statistics

We provide two simple formula to describe *BEG* distribution depending as to whether the parameter $b > 0$ is real non-integer or integer. First, if $|z| < 1$ and $b > 0$ is real non-integer thus we have

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} z^j. \tag{2.1}$$

Using the expansion (2.1) in (1.3), the cdf of the *BEG* distribution when $b > 0$ is real non-integer is

$$\begin{aligned} F(x) &= \frac{1}{B(a,b)} \int_0^{1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}} w^{a-1}(1-w)^{b-1} dw \\ &= \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \int_0^{1-\left\{1-\exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j!} w^{a+j-1} dw, \end{aligned}$$

and then

$$\begin{aligned} F(x) &= \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j) j! (a+j)} \\ &\quad \times \left(1 - \left\{1 - \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}\right)^{(a+j)}. \end{aligned} \tag{2.2}$$

Second, if $b > 0$ is integer, on applying the binomial expansion in (1.3) we have

$$\begin{aligned} F(x) &= \frac{1}{B(a,b)} \sum_{j=0}^{b-1} \frac{(-1)^j}{(a+j)} \\ &\quad \times \left(1 - \left\{1 - \exp\left[-\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}^{\alpha}\right)^{(a+j)}. \end{aligned} \tag{2.3}$$

Using the series

$$(1+z)^{\alpha} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+j-1)j!} z^{j}, \tag{2.4}$$

density function (1.4) can be expressed in the mixture form

$$f(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} w_{j,k} \exp\left(-\frac{x-\mu}{\sigma}\right) \exp\left[-(k+1)\exp\left(-\frac{x-\mu}{\sigma}\right)\right], \tag{2.5}$$

where $w_{j,k} = \dfrac{\alpha\Gamma(\alpha)\Gamma(\alpha(j+1)+b-1)}{\sigma\Gamma(\alpha-j)\Gamma(\alpha(j+1)+b-k-1)j!k!}$.

We now give the density function of the $i$th order statistics $X_{(i:n)}$, $f_{(i:n)}$ say, in a random sample of size $n$ from the $BEG$ distribution. It is well known that

$$f_{(i:n)} = \frac{1}{B(i,n-i+1)} f(x) F^{i-1}(x)\{1-F(x)\}^{n-i}, \tag{2.6}$$

for $i = 1, 2, ..., n$.

Using the expansion $(\sum_{i=0}^{\infty} a_i)^k$ for $a_{(i)}$,s $i = 0, 1,$ are real number, $k$ a positive integer and $m_r = 0, 1, ...$ $\{r = 1, 2, ..., k\}$ when $b > 0$ is real non-integer

$$f_{(i:n)}(x) = \sum_{k=0}^{n-i}\sum_{m_1=0}^{\infty} \cdots \sum_{m_{k+i-1}=0}^{\infty} \delta_{k,i}^{(1)} f_{(k,i)}(x), \tag{2.7}$$

and for $b > 0$ integer

$$f_{(i:n)}(x) = \sum_{k=0}^{n-i}\sum_{m_1=0}^{b-1} \cdots \sum_{m_{k+i-1}=0}^{b-1} \delta_{k,i}^{(2)} f_{(k,i)}(x). \tag{2.8}$$

Letting $f_{(k,i)}(x)$ represent the density of a random variable $X_{(k,i)}$ following a $BEG(2a + \sum_{r=1}^{k+i-1} m_r, b, \mu, \sigma, \alpha)$ distribution, the functions $\delta_{k,i}^{(1)}$ and $\delta_{k,i}^{(2)}$ required for the above expressions are

$$\begin{aligned}\delta_{k,i}^{(1)} &= \frac{B(2a+\sum_{r=1}^{k+i-1} m_r, b)}{B(i,n-i+1)} \frac{(-1)^{k+\sum_{r=1}^{k+i-1} m_r}}{[B(a,b)]^{k+1}} \\ &\times \frac{(\Gamma(b))^{k+i-1}}{\prod_{r=1}^{k+i-1}\Gamma(b-m_r)m_r!(a+m_r)}.\end{aligned}$$

and

$$\delta_{k,i}^{(2)} = \binom{n-i}{k}\frac{B(2a+\sum_{r=1}^{k+i-1} m_r, b)}{B(i, n-i+1)}\frac{(-1)^{k+\sum_{r=1}^{k+i-1} m_r}}{[B(a,b)]^{k+1}} \times \prod_{r=1}^{k+i-1}\frac{\binom{b-1}{m_r}}{(a+m_r)}.$$

## 3 Moments

The $r$th moment of the $X \sim BEG(a, b, \mu, \sigma, \alpha)$ can be written as

$$E(X^r) = \int_0^\infty \sum_{j=0}^\infty \sum_{k=0}^\infty w_{j,k} x^r \exp(-\frac{x-\mu}{\sigma}) \exp[-(k+1)\exp(-\frac{x-\mu}{\sigma})]dx,$$

which on setting $u = \exp\left(-\frac{x-\mu}{\sigma}\right)$, reduces to

$$E(X^r) = \sum_{j=0}^\infty \sum_{k=0}^\infty w_{j,k} \int_0^\infty \sigma(\mu - \sigma\log(u))^r \exp[-(k+1)u]du. \quad (3.1)$$

Using the binomial expansion Equation (3.1) can be written as

$$E(X^r) = \sigma \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^r \binom{r}{l} w_{j,k} \mu^{r-l}(-\sigma)^l I(l),$$

where $I(l)$ denotes the integral

$$I(l) = \int_0^\infty (\log(u))^l \exp[-(k+1)u]du. \quad (3.2)$$

Finally, by Equation (2.6.21.1) in Prudnikov et al. (1986, volume 1), Equation (3.2) can be calculated as

$$I(l) = \left(\frac{\partial}{\partial a}\right)^l [(k+1)^{-a}\Gamma(a)]|_{a=1}.$$

Then we have

$$E(X^r) = \sigma \sum_{j=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^r \binom{r}{l} w_{j,k} \mu^{r-l}(-\sigma)^l \left(\frac{\partial}{\partial a}\right)^l [(k+1)^{-a}\Gamma(a)]|_{a=1}. \quad (3.3)$$

We can easily obtain expression for the moment of order statistics from Equation (16). The $r$th moment of $X_{(i:n)}$ for $b > 0$ real non-integer is

$$E(X^r_{(i:n)}) = \sum_{k=0}^{n-i} \sum_{m_1=0}^{\infty} \dots \sum_{m_{k+i-1}=0}^{\infty} \delta^{(1)}_{k,i} E(X^r),$$

and for $b > 0$ integer

$$E(X^r_{(i:n)}) = \sum_{k=0}^{n-i} \sum_{m_1=0}^{b-1} \dots \sum_{m_{k+i-1}=0}^{b-1} \delta^{(2)}_{k,i} E(X^r),$$

where $X \sim BEG(2a + \sum_{r=1}^{k+i-1} m_r, b, \mu, \sigma, \alpha)$ distribution

## 4 Estimation and Inference

Let us assume that $Y$ follows the $BEG$ distribution and let $\theta = (a, b, \mu, \sigma, \alpha)^T$ be the parameter vector. The log-likelihood for a single observation $y$ of $Y$ is

$$\begin{aligned} \ell = \ell(a, b, \mu, \sigma, \alpha) \quad &= \quad \log(a) - \log(\sigma) - \log(B(a, b)) + \log(u) \\ &\quad -u + (\alpha b - 1)\log(1 - e^{-u}) \\ &\quad +(a - 1)\log[1 - (1 - e^{-u})^{\alpha}], \quad y > 0 \end{aligned}$$

where $u = e^{-\dfrac{(x-\mu)}{\sigma}}$.

The components of the unit score vector $U = (\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \mu}, \frac{\partial \ell}{\partial \sigma}, \frac{\partial \ell}{\partial \alpha})^T$ are

$$
\begin{aligned}
\frac{\partial \ell}{\partial a} &= \psi(a+b) - \psi(a) + \log[1-(1-e^{-u})^{\alpha}], \\
\frac{\partial l}{\partial b} &= \psi(a+b) - \psi(b) + \alpha \log(1-e^{-u}), \\
\frac{\partial \ell}{\partial \mu} &= \frac{1}{\sigma} - \frac{u}{\sigma} + \frac{\alpha b - 1}{\sigma} \frac{u e^{-u}}{1-e^{-u}} - \frac{\alpha(a-1)}{\sigma} \frac{u e^{-u}(1-e^{-u})^{\alpha-1}}{1-(1-e^{-u})^{\alpha}}, \\
\frac{\partial \ell}{\partial \sigma} &= -\frac{1}{\sigma} - \frac{\log(u)}{\sigma} + \frac{u \log(u)}{\sigma} - \frac{\alpha b - 1}{\sigma} \frac{u \log(u e^{-u})}{1-e^{-u}} \\
&\quad + \frac{\alpha(a-1)}{\sigma} \frac{u \log(u e^{-u})(1-e^{-u})^{\alpha-1}}{1-(1-e^{-u})^{\alpha}}, \\
\frac{\partial \ell}{\partial \alpha} &= \frac{1}{\alpha} + b \log(1-e^{-u}) - (a-1) \frac{(1-e^{-u})^{\alpha} \log(1-e^{-u})}{1-(1-e^{-u})^{\alpha}}.
\end{aligned}
$$

The expected value of the score vector is zero and then

$$
\begin{aligned}
E(\log[1-(1-e^{-u})^{\alpha}]) &= \psi(a+b) - \psi(a), \\
E(\log(1-e^{-u})) &= \frac{\psi(a+b) - \psi(b)}{\alpha}.
\end{aligned}
$$

For a random sample $y = (y_1, ..., y_n)$ of size $n$ from $BEG(a, b, \mu, \sigma, \alpha)$, the total log-likelihood is $\ell_n = \ell_n(a, b, \mu, \sigma, \alpha) = \sum_{i=1}^{n} \ell^{(i)}$, where $\ell^{(i)}$ is the log-likelihood for the $i$th observation ($i = 1, \ldots, n$). The total score function is $U_n = \sum_{i=1}^{n} U^{(i)}$ , where $U^{(i)}$ is the score function $y_i$ and it has the form given for $i = 1, ..., n$. The *MLE* $\hat{\theta}$ of $\theta$ is obtain numerically from the nonlinear equation $U_n = 0$. For interval estimation and tests of hypotheses on the parameters in $\theta$ we obtain the $5 \times 5$ unit information matrix

$$
K = K(\theta) = \begin{bmatrix}
K_{a,a} & K_{a,b} & K_{a,\mu} & K_{a,\sigma} & K_{a,\alpha} \\
K_{a,b} & K_{b,b} & K_{b,\mu} & K_{b,\sigma} & K_{b,\alpha} \\
K_{a,\mu} & K_{b,\mu} & K_{\mu,\mu} & K_{\mu,\sigma} & K_{\mu,\alpha} \\
K_{a,\sigma} & K_{b,\sigma} & K_{\mu,\sigma} & K_{\sigma,\sigma} & K_{\sigma,\alpha} \\
K_{a,\alpha} & K_{b,\alpha} & K_{\mu,\alpha} & K_{\sigma,\alpha} & K_{\alpha,\alpha}
\end{bmatrix}
$$

where the corresponding elements are given by

$$
K_{a,a} = \acute{\psi}(a) - \acute{\psi}(a+b), \quad K_{b,b} = \acute{\psi}(b) - \acute{\psi}(a+b), \quad K_{a,b} = -\acute{\psi}(a+b)
$$

$$
\begin{aligned}
K_{a,\mu} &= \frac{\alpha}{\sigma}T_{-1,1,-1,1,1,0,0}, \quad K_{a,\sigma} = -\frac{\alpha}{\sigma}T_{-1,1,-1,1,1,1,0}, \\
K_{a,\alpha} &= -\frac{1}{\alpha}T_{-1,0,1,0,0,0,1}, \quad K_{b,\mu} = \frac{\alpha}{\sigma}T_{0,0,-1,1,1,0,0}, \\
K_{b,\sigma} &= -\frac{\alpha}{\sigma}T_{0,0,-1,1,1,1,0}, \quad K_{b,\alpha} = T_{0,0,0,0,0,0,1}, \\
K_{\mu,\mu} &= \frac{1}{\sigma^2}(-T_{0,0,0,0,1,0,0} + (\alpha b-1)(T_{0,0,-1,1,1,0,0} - T_{0,0,2,1,2,0,0})) \\
&- \frac{\alpha(a-1)}{\sigma}T_{-2,0,1,0,0,0,0} - T_{-2,0,1,0,1,0,0} \\
&+ (\alpha-1)T_{-1,0,0,1,1,0,0} + \alpha T_{-2,2,-2,2,2,0,0}, \\
K_{\mu,\sigma} &= -\frac{1}{\sigma^2}(-T_{0,0,0,0,1,1,0} + (\alpha b-1)(T_{0,0,-1,1,1,1,0} - T_{0,0,2,1,2,1,0})) \\
&- \frac{1}{\sigma^2}\frac{\alpha(a-1)}{\sigma}(T_{-2,0,-1,0,0,1,0} - T_{-2,0,1,0,1,1,0} + (\alpha-1)T_{-1,0,0,1,1,1,0}) \\
&+ \frac{\alpha}{\sigma^2}\frac{\alpha(a-1)}{\sigma}T_{-2,2,-2,2,2,1,0} - \frac{1}{\sigma^2}(1 - T_{0,0,0,0,1,0,0}) \\
&- \frac{1}{\sigma^2}((\alpha b-1)T_{0,0,-1,1,1,0,0} - \alpha(a-1)T_{-1,1,-1,1,1,0,0}), \\
K_{\mu,\alpha} &= \frac{1}{\sigma}(bT_{0,0,-1,1,1,0,0} - (a-1)T_{-1,0,1,1,1,0,0}) \\
&- \frac{\alpha}{\sigma}(a-1)(T_{-1,1,-1,1,1,0,1} + T_{-2,2,0,1,1,0,0})
\end{aligned}
$$

$$
\begin{aligned}
K_{\sigma,\sigma} &= \frac{1}{\sigma}T_{0,0,0,0,0,1,0} - \frac{1}{\sigma^2}T_{0,0,0,0,1,2,0} - \frac{1}{\sigma^2}T_{0,0,0,0,1,1,0} \\
&+ \frac{(\alpha b-1)}{\sigma^2}(T_{0,0,-1,1,1,2,0} + T_{0,0,-1,1,1,1,0} + T_{0,0,-2,1,2,2,0}) \\
&- \frac{\alpha(a-1)}{\sigma^2}(T_{-1,1,-1,1,2,2,0} + T_{-1,1,-1,1,1,2,0} + T_{-1,1,-1,1,1,1,0}) \\
&- \frac{\alpha(a-1)^2}{\sigma^2}(T_{-1,1,-2,2,2,2} + T_{-2,0,2,2,2,0}) \\
&+ \frac{1}{\sigma^2}(1 - T_{0,0,0,0,0,1,0} + T_{0,0,0,0,1,1,0} - (\alpha b-1)T_{0,0,0,1,1,1,0}) \\
&+ \frac{\alpha(a-1)}{\sigma^2}T_{-1,1,-1,1,1,1,0},
\end{aligned}
$$

$$
\begin{aligned}
K_{\sigma,\alpha} &= \frac{1}{\sigma}(bT_{0,0,-1,1,1,1,0} - (a-1)T_{-1,1,-1,1,1,1,0}) \\
&- \frac{\alpha}{\sigma}(a-1)(T_{-1,1,-1,1,1,1,1} + T_{-2,2,0,1,1,1,0}), \\
K_{\alpha,\alpha} &= -\frac{1}{\alpha^2} + (a-1)(T_{-1,1,0,0,0,0,2} + T_{-2,2,0,0,0,0,2}).
\end{aligned}
$$

Here, we have defined the following expectation

$$
\begin{aligned}
T_{i,j,k,l,m,r,q} &= E[V^i(1-V)^{j+\frac{k}{\alpha}}(1-(1-V)^{\frac{1}{\alpha}})^l(\log(1-(1-V)^{\frac{1}{\alpha}}))^m \\
&\times (\log(-\log(1-(1-V)^{\frac{1}{\alpha}})))^r(\frac{1}{\alpha}\log(1-V))^q]; \\
&\quad i,j,k,l,m,r,q \in \{-2,-1,0,1,2\}
\end{aligned}
$$

where $V \sim Beta(a,b)$ and the integral obtained from the above definition is numerically determined using MATLAB for any $a,b$ and $\alpha$. For example, for $a = 0.9144, b = 0.9991$ and $\alpha = 0.7910$ we easily calculated some $T$,s in the information matrix:

$$
\begin{aligned}
T_{-1,1,-1,1,1,0,0} &= -1.2483, T_{-1,0,1,0,0,0,1} = -0.7258, T_{0,0,2,1,2,0,0} = 0.1215, \\
T_{-1,1,-1,1,1,0,1} &= 0.4834, T_{-1,1,0,0,0,0,2} = 0.6214, \quad T_{-2,2,0,0,0,0,2} = 0.8070.
\end{aligned}
$$

The total information matrix is then $K_n = K_n(\theta) = nK(\theta)$.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of

$$\sqrt{n}(\hat{\theta} - \theta)$$

is $N_5(0, K(\theta^{-}1))$. The asymptotic multivariate normal $N_5(0, K(\theta^{-}1))$ distribution of $\hat{\theta}$ can be used to construct approximate confidence intervals and confidence regions for the parameters and for the hazard and survival functions. The asymptotically normality is also useful for testing goodness of fit of the *BEG* distribution and for comparing this distribution with some of its special sub-models using one of the three well-known asymptotically equivalent test statistics namely, the likelihood ratio (*LR*) statistic, Rao score ($S_R$) and Wald ($W$) statistics.

An asymptotic confidence interval with significance level $\gamma$ for each parameter $\theta_i$ is given by

$$ACI(\theta_i, 100(1-\gamma)\%) = (\hat{\theta}_i - z_{\gamma/2}\sqrt{k^{\theta_i,\theta_i}}, \hat{\theta}_i + z_{\gamma/2}\sqrt{k^{\theta_i,\theta_i}}),$$

where $k^{\theta_i,\theta_i}$ is the $i$th diagonal element of $K_n(\theta)^{-1}$ for $i = 1,2,3,4,5$ and $z_{\gamma/2}$ is the quantile $1-\gamma/2$ of the standard normal distribution.

Further, we can compute the maximum values of the unrestricted and restricted log-likelihoods to construct the $LR$ statistics for testing some sub-models of the $BEG$ distribution.

We consider the partition $\theta = (\theta_1^T, \theta_2^T)^T$, where $\theta_1$ is a subset of the parameters of interest of the $BEG$ and $\theta_2$ is a subset of the remaining parameters. The $LR$ statistic for testing the null hypothesis $H_0 : \theta_1 = \theta_1^{(0)}$ versus the alternative hypothesis $H_1 : \theta_1 \neq \theta_1^{(0)}$ is given by $w = 2\{\ell(\hat{\theta}) - \ell(\tilde{\theta})\}$, where $\tilde{\theta}$ and $\hat{\theta}$ denote the $MLE$s under the null and alternative hypothesis, respectively. The statistic $w$ is asymptotically (as $n \to \infty$) distributed as $\chi_k^2$, where $k$ is the dimension of the subset $\theta_1$ of interest.

## 5 Application

In this section we fit the $BEG$ distribution to example of real data and test three types of hypotheses $H_0 : Gumbel \times H_1 : BEG$, $H_0 : EG \times H_1 : BEG$ and $H_0 : BG \times H_1 : BEG$. The data are the annual maximum daily rainfall in Sweden for location Stockholm. The data set is:25.5, 40, 22.8, 38.8, 27, 43, 33.9, 31.9, 36.5, 22.4, 25.6, 35.8, 23.4, 41.1, 30.9, 28.4, 39.7, 56, 32.3, 49.8, 26, 23.6, 21.7, 44.9, 20.8, 31, 18.2, 54.1, 27.8, 26, 25, 45.8, 40.4, 31, 31.7, 34.6, 14.5, 23.7, 29.5, 23.3, 24.2, 24, 20.5, 32.2, 27.6, 59.8.
The $MLE$s and the maximized log-likelihood using the $BEG$ distribution are

$$\hat{a} = 0.9144, \quad \hat{b} = 0.9991, \quad \hat{\mu} = 26.4636, \quad \hat{\sigma} = 6.6051, \quad \hat{\alpha} = 0.7910,$$
$$\hat{\ell}_{BEG} = -174.8821,$$

whereas for the $BG$ , $EG$ and Gumbel distribution we obtain

$$\hat{a} = 1.69, \quad \hat{b} = 0.62, \quad \hat{\mu} = 16.53, \quad \hat{\sigma} = 6.44, \quad hat\ell_{BG} = -179.6892,$$
$$\hat{\mu} = 25.8849, \quad \hat{\sigma} = 6.6365, \quad \hat{\alpha} = 0.8015, \quad \hat{\ell}_{EG} = -184.0968,$$

and

$$\hat{\mu} = 27.2814, \quad \hat{\sigma} = 7.5667, \quad \hat{\ell}_{Gumbel} = -184.1654,$$

respectively.

For the data set, the values of the *LR* statistics for testing the hypotheses $H_0$ : *Gumbel* $\times$ $H_1$ : *BEG*, $H_0$ : *EG* $\times$ $H_1$ : *BEG* and $H_0$ : *BG* $\times$ $H_1$ : *BEG* are: 18.5666 ($p - value = 3.3601 \times 10^{-4}$), 18.4294 ($p - value = 9.9567 \times 10^{-5}$) and 9.6143 ($p - value = 19 \times 10^{-4}$), respectively. Therefore, we reject the null hypotheses in three cases in favor of the *BEG* distribution at the significance level of 5%. The plots of the estimated densities of the *BEG*, *BG*, *EG* and Gumbel distributions fitted to the data set given in Figure 2 show that the *BEG* distribution gives a better fit than the other three submodels.

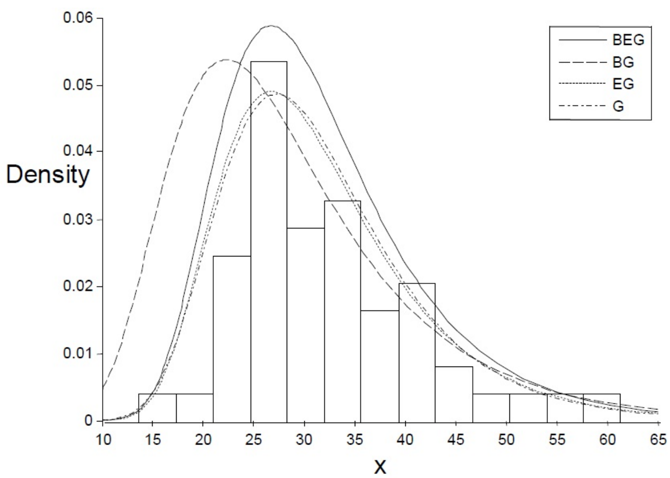



Figure 1: Estimated densities of the BEG, BG, EG and Gumbel distributions for the data set

## 6 Concluding Remarks

We proposed a new five-parameter distribution called the beta exponentiated Gumbel (*BEG*) distribution that includes the beta Gumbel, exponentiated Gumbel and Gumbel distribution. We obtained expressions for the distribution function, density and *r*th moment of the new distribution and order statistics. We discuss estimation of the parameters by maximum likelihood and provide the information matrix. We observe in one application to real data set that the *BEG* distribution flexible and can be used quite effectively in analysing positive data in place of the special cases.

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