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Integral Properties of Zonal Spherical Functions, Hypergeometric Functions and Invariant Polynomials

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Abstract. Some integral properties of zonal spherical functions, hypergeometric functions and invariant polynomials are studied for real normed division algebras.

Keywords. Division algebras, generalised beta and gamma functions, generalised hypergeometric functions, invariant polynomials, Jack polynomials, real, complex, quaternion and octonion random matrices, spherical functions on symmetric cones, zonal polynomials.

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1 Introduction

During the 1960s, real and complex zonal polynomials were studied exhaustively by [28, 29], [6] and [31], among many others. Excellent reference books include those by [37], [41], [19] and [35], which summarise many of the results published to date.

Hypergeometric functions with a matrix argument were first studied by [27] and defined in terms of zonal polynomials by [6]. Hypergeometric functions of one or two matrix arguments have been applied in many areas of science and technology, including multivariate statistical

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analysis ([37] and [35]), random matrix theory ([36] and [20]), wireless communications ([38, 39], shape theory [21] and [3]).

Later [7], [8], [4] and [5], introduced a class of homogeneous invariant polynomials with two or more matrix arguments, which generalise the zonal polynomials; many of their basic and integral properties are studied in real cases.

In the context of multivariate statistics, zonal polynomials were initially used to express many noncentral matrix variate distributions. However, there were other distributional problems that could not be solved using zonal polynomials. In these latter cases, invariant polynomials were used to obtain explicit expressions of doubly noncentral matrix variate distributions, matrix variate distribution functions and the joint density of eigenvalues of matrix variate beta type I and II distributions, etc. see [29] and [8].

During the 1980s and 1990s, zonal polynomials regained prominence but from a more general point of view, in which it was observed that zonal polynomial for real and complex cases are particular cases of Jack polynomials, see [40] and [22], among many others. In terms of Jack polynomials, it is possible to give a general definition for hypergeometric functions, see [23], [24], [15], [33] and [2], among many others. The Jack polynomials and their corresponding hypergeometric functions depend on a parameter $\alpha = 2/\beta$ that, only for the values $\beta = 1, 2, 4, 8$ and hypercomplex cases is interpreted from the matrix space. In this particular case, the Jack polynomials are termed real ($\beta = 1$), complex ($\beta = 2$), quaternion ($\beta = 4$), octonion ($\beta = 8$) and hypercomplex zonal (2β) polynomials, zonal spherical functions or spherical functions for symmetric cones, see [29], [34], [30] and [40], among others. In the rest of this work we shall adopt the nomenclature *zonal spherical functions* to refer to Jack polynomials in the particular cases considered here.

The properties for Jack polynomials and hypergeometric functions with a matrix argument have been studied by [27], [29], [6], [31] and [37] in the real case (zonal polynomials); by [29], [41], [19] and [38, 39] in the complex case (Schur functions); by [34] in the quaternion case and by [23], [33], [40] and [2] in the general case (real, complex and quaternion cases), among many others.

A serious obstacle encountered when Jack polynomials, hypergeometric functions and invariant polynomials are to be used is the question of their calculation. Fortunately, with the excellent algorithm proposed by [33] and [32], it is now possible to use these techniques in many applications, see [39] and [3]. Unfortunately, this obstacle remains for the general case of invariant polynomials.

The study of the Jack polynomials, hypergeometric functions and invariant polynomials in the octonion case is at present only of theoretical interest. Furthermore, according to [1], there is still no proof that octonions are useful for understanding the real world. Still, some other generalisations in hypercomplex cases are proposed, but as the author explains, this work only has theoretical interest, for the moment; see [30].

In this paper, we derive several integral properties of zonal spherical functions, hypergeometric functions and invariant polynomials for real normed division algebras, supplementing the work done by [23]. As an example, some results and their proofs are included. The proofs of the other results given in this paper are omitted, either because they follow immediately from the proven results or because these proofs can be derived along the same lines as the ones given for the real or complex cases. This is done in full awareness that the article might thus appear a mere compilation of formulae, but our aim is to prevent it from becoming overly lengthy. Note that we can only conjecture the results for the octonion case, because many of its related matrix problems are still under study. However, for example in [20, Section 1.4.5, pp. 22-24] it is proved that the bi-dimensional density function of the eigenvalue, for a 2×2 octonionic matrix with symmetric normal distribution, is obtained from the general joint density function of the eigenvalues for the symmetric normal distribution, assuming $m = 2$ and $\beta = 8$. The material in the present paper is organised as follows: Section 2 provides some notation and preliminary results about real normed division algebras, Jacobians, gamma and beta multivariate functions and invariant measures. Several integral properties of zonal spherical functions are obtained in Section 3. Many extensions of the integral properties of hypergeometric functions with one and two arguments are studied in Section 4. For invariant polynomials with two matrix arguments, in Section 5 we derive diverse integral properties, such as the inverse Laplace transformation, gamma and beta integrals, etc. Finally, in Section 6, we show diverse applications of some results derived previously, such as the distribution function of a central Wishart distribution for normed division algebras, its joint eigenvalue density and the distribution function of the largest and smallest eigenvalues. We emphasise the conditions that must be met by the parameters that take part in many integral properties in the cases discussed, because, even in the original references, these conditions were omitted or established inexactly.

2 Preliminary

A detailed discussion of real normed division algebras may be found in [1] and [16]. For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes: Let $\mathbb{F}$ be a field. An *algebra* $\mathfrak{F}$ over $\mathbb{F}$ is a pair $(\mathfrak{F}; m)$, where $\mathfrak{F}$ is a *finite-dimensional vector space* over $\mathbb{F}$ and *multiplication* $m : \mathfrak{F} \times \mathfrak{F} \to A$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}$, $x, y, z \in \mathfrak{F}$;

$$\begin{aligned} m(x, \lambda y + z) &= \lambda m(x; y) + m(x; z) \\ m(\lambda x + y; z) &= \lambda m(x; z) + m(y; z). \end{aligned}$$

Two algebras $(\mathfrak{F}; m)$ and $(\mathfrak{E}; n)$ over $\mathbb{F}$ are said to be *isomorphic* if there is an invertible map $\phi : \mathfrak{F} \to \mathfrak{E}$ such that for all $x, y \in \mathfrak{F}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{F}$.

Let $\mathfrak{F}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{F}$ is said to be

1. *alternative* if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{F}$,
2. *associative* if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{F}$,
3. *commutative* if $xy = yx$ for all $x, y \in \mathfrak{F}$, and
4. *unital* if there is a $1 \in \mathfrak{F}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{F}$.

If $\mathfrak{F}$ is unital, then the identity 1 is uniquely determined.

An algebra $\mathfrak{F}$ over $\mathbb{F}$ is said to be a *division algebra* if $\mathfrak{F}$ is nonzero and $xy = 0_{\mathfrak{F}} \Rightarrow x = 0_{\mathfrak{F}}$ or $y = 0_{\mathfrak{F}}$ for all $x, y \in \mathfrak{F}$.

The term “division algebra”, comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathfrak{F}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{F}$ is a division algebra if, and only if, $\mathfrak{F}$ is nonzero and for all $a, b \in \mathfrak{F}$, with $b \neq 0_{\mathfrak{F}}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathfrak{F}$.

In the sequel we assume $\mathbb{F} = \Re$ and consider classes of division algebras over $\Re$ or “*real division algebras*” for short.

We introduce the algebras of *real numbers* $\Re$, *complex numbers* $\mathfrak{C}$, *quaternions* $\mathfrak{H}$ and *octonions* $\mathfrak{O}$. Then, if $\mathfrak{F}$ is an alternative real division algebra, then $\mathfrak{F}$ is isomorphic to $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

Let $\mathfrak{F}$ be a real division algebra with identity 1. Then $\mathfrak{F}$ is said to be *normed* if there is an inner product $(\cdot,\cdot)$ on $\mathfrak{F}$ such that

$$(xy, xy) = (x, x)(y, y) \qquad \text{for all } x, y \in \mathfrak{F}.$$

If $\mathfrak{F}$ is a *real normed division algebra*, then $\mathfrak{F}$ is isomorphic to $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

There are exactly four normed division algebras: real numbers ($\Re$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see [1]. We take into account that, $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ and $\mathfrak{O}$ are the only normed division algebras; furthermore, they are the only alternative division algebras.

Let $\mathfrak{F}$ be a division algebra over the real numbers. Then $\mathfrak{F}$ has dimension either 1, 2, 4 or 8. In other branches of mathematics, the parameters $\alpha = 2/\beta$ and $t = \beta/4$ are used, see [17] and [30], respectively.

Table 1: Values of $\beta = 2/\alpha$ and $t = \beta/4$ parameters.

β	α	t	Normed divison algebra
1	2	1/4	real ($\Re$)
2	1	1/2	complex ($\mathfrak{C}$)
4	1/2	1	quaternionic ($\mathfrak{H}$)
8	1/4	2	octonion ($\mathfrak{O}$)

Finally, observe that

$\Re$ is a real commutative associative normed division algebras,
$\mathfrak{C}$ is a commutative associative normed division algebras,
$\mathfrak{H}$ is an associative normed division algebras,
$\mathfrak{O}$ is an alternative normed division algebras.

Let $\mathcal{L}^{\beta}_{m,n}$ be the set of all $n \times m$ matrices of rank $m \le n$ over $\mathfrak{F}$ with m distinct positive singular values, where $\mathfrak{F}$ denotes a *real finite-dimensional normed division algebra*. In particular, let $GL(m,\mathfrak{F})$ be the space of all invertible $m \times m$ matrices over $\mathfrak{F}$. Let $\mathfrak{F}^{n\times m}$ be the set of all $n \times m$ matrices over $\mathfrak{F}$. The dimension of $\mathfrak{F}^{n\times m}$ over $\Re$ is βmn.

Let $\mathbf{A} \in \mathfrak{F}^{n\times m}$, then $\mathbf{A}^* = \overline{\mathbf{A}}^T$ denotes the usual conjugate transpose. The set of matrices $\mathbf{H}_1 \in \mathfrak{F}^{n\times m}$ such that $\mathbf{H}_1^*\mathbf{H}_1 = \mathbf{I}_m$ is a manifold denoted $\mathcal{V}^{\beta}_{m,n}$, termed the *Stiefel manifold* ($\mathbf{H}_1$ are also known as *semi-orthogonal* ($\beta = 1$), *semi-unitary* ($\beta = 2$), *semi-symplectic* ($\beta = 4$) and *semi-exceptional type* ($\beta = 8$) matrices, see [13]). The dimension of $\mathcal{V}^{\beta}_{m,n}$ over $\Re$ is $[\beta mn - m(m-1)\beta/2 - m]$. In particular, $\mathcal{V}^{\beta}_{m,m}$ with dimension over $\Re$, $[m(m+1)\beta/2 - m]$, is the maximal compact subgroup $\mathfrak{U}^\beta(m)$ of $\mathcal{L}^\beta_{m,m}$ and consists of all matrices $\mathbf{H} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{H}^*\mathbf{H} = \mathbf{I}_m$. Therefore, $\mathfrak{U}^\beta(m)$ is the *real orthogonal group* $\mathcal{O}(m)$ ($\beta = 1$), the *unitary group* $\mathcal{U}(m)$ ($\beta = 2$), the *compact symplectic group* $\mathcal{S}p(m)$ ($\beta = 4$) or *exceptional type matrices* $\mathcal{O}o(m)$ ($\beta = 8$), for $\mathfrak{F} = \Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$, respectively. Denote by $\mathfrak{S}^\beta_m$ the real vector space of all $\mathbf{S} \in \mathfrak{F}^{m\times m}$ such that $\mathbf{S} = \mathbf{S}^*$. Let $\mathfrak{P}^\beta_m$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{F}^{m\times m}$. Thus, $\mathfrak{P}^\beta_m$ consist of all matrices $\mathbf{S} = \mathbf{X}^*\mathbf{X}$, with $\mathbf{X} \in \mathfrak{L}^\beta_{m,n}$; then $\mathfrak{P}^\beta_m$ is an open subset of $\mathfrak{S}^\beta_m$. Over $\Re$, $\mathfrak{S}^\beta_m$ consist of *symmetric* matrices; over $\mathfrak{C}$, *Hermitian* matrices; over $\mathfrak{H}$, *quaternionic Hermitian* matrices (also termed *self-dual matrices*) and over $\mathfrak{O}$, *octonionic Hermitian* matrices. Generically, the elements of $\mathfrak{S}^\beta_m$ are termed as **Hermitian matrices**, irrespective of the nature of $\mathfrak{F}$. The dimension of $\mathfrak{S}^\beta_m$ over $\Re$ is $[m(m-1)\beta + 2m]/2$. Let $\mathfrak{D}^\beta_m$ be the *diagonal subgroup* of $\mathcal{L}^\beta_{m,m}$ consisting of all $\mathbf{D} \in \mathfrak{F}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$. Let $\mathfrak{T}^\beta_L(m)$ be the subgroup of all *lower triangular* matrices $\mathbf{T} \in \mathfrak{F}^{m\times m}$ such that $t_{ij} = 0$ for $1 < i < j \leq m$; and let $\mathfrak{T}^\beta_U(m)$ be the opposed *upper triangular* subgroup $\mathfrak{T}^\beta_U(m) = \left(\mathfrak{T}^\beta_L(m)\right)^T$. For any matrix $\mathbf{X} \in \mathfrak{F}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{F}^{m\times n}, \mathfrak{S}^\beta_m$, $\mathfrak{D}^\beta_m$ or $\mathcal{V}^\beta_{m,n}$, see [14].

If $\mathbf{X} \in \mathfrak{F}^{n\times m}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{F}^{n\times m}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m}\bigwedge_{k=1}^{\beta} dx^{(k)}_{ij}.$$

Remark 2.1. *Note that for $x_{ij} \in \mathfrak{F}$*

$$dx_{ij} = \bigwedge_{k=1}^{\beta} dx^{(k)}_{ij}.$$

In particular for $\mathfrak{F} = \Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$ we have

- $x_{ij} \in \Re$ *then*

$$dx_{ij} = \bigwedge_{k=1}^{1} dx^{(k)}_{ij} = dx_{ij}.$$

- $x_{ij} = x^{(1)}_{ij} + ix^{(2)}_{ij} \in \mathfrak{C}$, *then*

$$dx_{ij} = dx^{(1)}_{ij} \wedge dx^{(2)}_{ij} = \bigwedge_{k=1}^{2} dx^{(k)}_{ij}.$$

- $x_{ij} = x_{ij}^{(1)} + ix_{ij}^{(2)} + jx_{ij}^{(3)} + kx_{ij}^{(4)} \in \mathfrak{H}$, *then*

$$dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} \wedge dx_{ij}^{(3)} \wedge dx_{ij}^{(4)} = \bigwedge_{k=1}^{4} dx_{ij}^{(k)}.$$

- $x_{ij} = x_{ij}^{(1)} + e_1 x_{ij}^{(2)} + e_2 x_{ij}^{(3)} + e_3 x_{ij}^{(4)} + e_4 x_{ij}^{(5)} + e_5 x_{ij}^{(6)} + e_6 x_{ij}^{(7)} + e_7 x_{ij}^{(8)} \in \mathfrak{O}$, *then*

$$dx_{ij} = dx_{ij}^{(1)} \wedge dx_{ij}^{(2)} \wedge dx_{ij}^{(3)} \wedge dx_{ij}^{(4)} \wedge dx_{ij}^{(5)} \wedge dx_{ij}^{(6)} \wedge dx_{ij}^{(7)} \wedge dx_{ij}^{(8)} = \bigwedge_{k=1}^{8} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}_m^\beta$ (or $\mathbf{S} \in \mathfrak{T}_U^\beta(m)$) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}_m^\beta$ or in $\mathfrak{T}_U^\beta(m)$) denotes the exterior product of the $m(m-1)\beta/2 + m$ functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m} \bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Observe, that for the Lebesgue measure $(d\mathbf{S})$ defined thus, it is required that $\mathbf{S} \in \mathfrak{P}_m^\beta$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian positive definite matrix). In the real case, when $\mathbf{S}$ is a positive semidefinite matrix, its corresponding measure is studied in [43], [11], [9] and [10] under different coordinate systems.

If $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$ then $(d\mathbf{\Lambda})$ (the Lebesgue measure in $\mathfrak{D}_m^\beta$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{m} \bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

If $\mathbf{H}_1 \in \mathcal{V}_{m,n}^\beta$ then

$$(\mathbf{H}_1^* d\mathbf{H}_1) = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{m} \mathbf{h}_j^* d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) = (\mathbf{h}_1, \dots, \mathbf{h}_m|\mathbf{h}_{m+1}, \dots, \mathbf{h}_n) \in \mathfrak{U}^\beta(n)$. It can be proved that this differential form does not depend on the choice of the matrix $\mathbf{H}_2$ and that it is invariant under the transformations

$$\mathbf{H}_1 \to \mathbf{Q}\mathbf{H}_1\mathbf{P}, \quad \mathbf{Q} \in \mathfrak{U}^\beta(n) \text{ and } \mathbf{P} \in \mathfrak{U}^\beta(m). \tag{2.1}$$

When $m = 1$; $\mathcal{V}_{1,n}^{\beta}$ defines the unit sphere in $\mathfrak{F}^n$. This is, of course, an $(n-1)\beta$- dimensional surface in $\mathfrak{F}^n$. When $m = n$ and denoting $\mathbf{H}_1$ by $\mathbf{H}$, $(\mathbf{H}^* d\mathbf{H})$ is termed the *Haar measure* on $\mathfrak{U}^{\beta}(m)$ and defines an invariant differential form of a unique measure ν on $\mathfrak{U}^{\beta}(m)$ given by

$$\nu(\mathfrak{M}) = \int_{\mathfrak{M}} (\mathbf{H}^* d\mathbf{H}).$$

It is unique in the sense that any other invariant measure on $\mathfrak{U}^{\beta}(m)$ is a finite multiple of ν and invariant because it is invariant under left and right translations, that is

$$\nu(\mathbf{Q}\mathfrak{M}) = \nu(\mathfrak{M}\mathbf{Q}) = \nu(\mathfrak{M}), \quad \forall \mathbf{Q} \in \mathfrak{U}^{\beta}(m).$$

The surface area or volume of the Stiefel manifold $\mathcal{V}_{m,n}^{\beta}$ is

$$\operatorname{Vol}(\mathcal{V}_{m,n}^{\beta}) = \int_{\mathbf{H}_1 \in \mathcal{V}_{m,n}^{\beta}} (\mathbf{H}_1^* d\mathbf{H}_1) = \frac{2^m \pi^{mn\beta/2}}{\Gamma_m^{\beta}[n\beta/2]}, \tag{2.2}$$

and therefore

$$(d\mathbf{H}_1) = \frac{1}{\operatorname{Vol}\left(\mathcal{V}_{m,n}^{\beta}\right)} (\mathbf{H}_1^* d\mathbf{H}_1) = \frac{\Gamma_m^{\beta}[n\beta/2]}{2^m \pi^{mn\beta/2}} (\mathbf{H}_1^* d\mathbf{H}_1).$$

is the *normalised invariant measure on* $\mathcal{V}_{m,n}^{\beta}$ and $(d\mathbf{H})$, i.e. with $(m = n)$, defines the *normalised Haar measure* on $\mathfrak{U}^{\beta}(m)$. In (2.2), $\Gamma_m^{\beta}[a]$ denotes the *multivariate Gamma function* for the space $\mathfrak{S}_m^{\beta}$, and is defined by

$$\begin{aligned}
\Gamma_m^{\beta}[a] &= \int_{\mathbf{A} \in \mathfrak{P}_m^{\beta}} \operatorname{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} (d\mathbf{A}) \\
&= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - (i-1)\beta/2] \\
&= \pi^{m(m-1)\beta/4} \prod_{i=1}^{m} \Gamma[a - (m-i)\beta/2], \qquad (2.3)
\end{aligned}$$

where $\operatorname{etr}\{\cdot\} = \exp\{\operatorname{tr}(\cdot)\}$, $|\cdot|$ denotes the determinant and $\operatorname{Re}(a) > (m-1)\beta/2$, see [23]. This can be obtained as a particular case of the *generalised gamma function of weight* κ for the space $\mathfrak{S}_m^{\beta}$ with $\kappa =$

$(k_1, k_2, \ldots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, taking $\kappa = (0, 0, \ldots, 0)$ and which for $\mathrm{Re}(a) \geq (m-1)\beta/2 - k_m$ is defined by, see [23],

$$\begin{aligned}
\Gamma_m^\beta[a, \kappa] &= \int_{\mathbf{A} \in \mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{A}\} |\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A})(d\mathbf{A}) \\
&= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2] \\
&= \pi^{m(m-1)\beta/4} \prod_{i=1}^m \Gamma[a + k_i - (m-i)\beta/2] \\
&= [a]_\kappa^\beta \Gamma_m^\beta[a], \qquad (2.4)
\end{aligned}$$

where for $\mathbf{A} \in \mathfrak{S}_m^\beta$

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m} \prod_{i=1}^{m-1} |\mathbf{A}_i|^{k_i - k_{i+1}} \qquad (2.5)$$

with $\mathbf{A}_p = (a_{rs})$, $r, s = 1, 2, \ldots, p$, $p = 1, 2, \ldots, m$ is termed the *highest weight vector*, see [23].

Remark 2.2. *Let $\mathcal{P}(\mathfrak{S}_m^\beta)$ denote the algebra of all polynomial functions on $\mathfrak{S}_m^\beta$, and $\mathcal{P}_k(\mathfrak{S}_m^\beta)$ the subspace of homogeneous polynomials of degree k and let $\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$ be an irreducible subspace of $\mathcal{P}(\mathfrak{S}_m^\beta)$ such that*

$$\mathcal{P}_k(\mathfrak{S}_m^\beta) = \sum_\kappa \bigoplus \mathcal{P}^\kappa(\mathfrak{S}_m^\beta).$$

Note that q_κ is a homogeneous polynomial of degree k, moreover $q_\kappa \in \mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$, see [23].

In (2.4), $[a]_\kappa^\beta$ denotes the generalised Pochhammer symbol of weight κ, defined as

$$\begin{aligned}
[a]_\kappa^\beta &= \prod_{i=1}^m (a - (i-1)\beta/2)_{k_i} \\
&= \frac{\pi^{m(m-1)\beta/4} \displaystyle\prod_{i=1}^m \Gamma[a + k_i - (i-1)\beta/2]}{\Gamma_m^\beta[a]} \\
&= \frac{\Gamma_m^\beta[a, \kappa]}{\Gamma_m^\beta[a]},
\end{aligned}$$

where $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$ and

$$(a)_i = a(a+1)\cdots(a+i-1),$$

is the standard Pochhammer symbol.

A variant of the generalised gamma function of weight κ is obtained from [31] and is defined as

$$\begin{aligned}
\Gamma_m^\beta[a,-\kappa] &= \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}^{-1})(d\mathbf{A}) \\
&= \pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a-k_i-(m-i)\beta/2] \\
&= \pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a-k_i-(i-1)\beta/2] \\
&= \frac{(-1)^k\Gamma_m^\beta[a]}{[-a+(m-1)\beta/2+1]_\kappa^\beta}, \qquad (2.6)
\end{aligned}$$

where $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$.

The two expressions of $\Gamma_m^\beta[a,]$, $\Gamma_m^\beta[a,\kappa]$ and $\Gamma_m^\beta[a,-\kappa]$ as the product of ordinary gamma functions are obtained using the proofs corresponding to $\mathbf{A} = \mathbf{T}\mathbf{T}^*$ and $\mathbf{A} = \mathbf{T}^*\mathbf{T}$ with the corresponding Jacobian given in Proposition 2.2. Alternatively, note that for any function $g(y)$

$$\prod_{i=1}^q g(x+i-1) = \prod_{i=1}^q g(x+q-i), \qquad (2.7)$$

and

$$\prod_{i=1}^q g(x-i+1) = \prod_{i=1}^q g(x-q+i). \qquad (2.8)$$

Similarly, from [27, p. 480] the *multivariate beta function* for the space $\mathfrak{S}_m^\beta$, can be defined as

$$\begin{aligned}
\mathcal{B}_m^\beta[b,a] &= \int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1}|\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1}(d\mathbf{S}) \\
&= \int_{\mathbf{R}\in\mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1}|\mathbf{I}_m+\mathbf{R}|^{-(a+b)}(d\mathbf{R}) \\
&= \frac{\Gamma_m^\beta[a]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b]}, \qquad (2.9)
\end{aligned}$$

where $\mathbf{R} = (\mathbf{I}-\mathbf{S})^{-1}-\mathbf{I}$, $\mathrm{Re}(a) > (m-1)\beta/2$ and $\mathrm{Re}(b) > (m-1)\beta/2$.

Some Jacobians in the quaternionic case are obtained in [34]. We now cite some Jacobians in terms of the parameter β, based on the works of [30] and [14]. We also include a parameter count (or number of functionally independent variables, #fiv), that is, if $\mathbf{A}$ is factorised as $\mathbf{A} = \mathbf{BC}$, then the parameter count is written as #fiv in $\mathbf{A}$ = [#fiv in $\mathbf{B}$] + [#fiv in $\mathbf{C}$], see [14].

Proposition 2.1. *Let $\mathbf{X}$ and $\mathbf{Y} \in \mathfrak{S}_m^\beta$ be matrices of functionally independent variables, and let $\mathbf{Y} = \mathbf{AXA}^* + \mathbf{C}$, where $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$ and $\mathbf{C} \in \mathfrak{S}_m^\beta$ are matrices of constants. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^*\mathbf{A}|^{\beta(m-1)/2+1}(d\mathbf{X}). \tag{2.10}$$

Proposition 2.2. (Cholesky's decomposition) *Let $\mathbf{S} \in \mathfrak{P}_m^\beta$ and $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ with $t_{ii} > 0$, $i = 1, 2, \ldots, m$. Then*

- *parameter count: $\beta m(m-1)/2 + m$ and*

$$(d\mathbf{S}) = \begin{cases} 2^m \displaystyle\prod_{i=1}^m t_{ii}^{\beta(m-i)+1}(d\mathbf{T}) & \text{if } \mathbf{S} = \mathbf{T}^*\mathbf{T}; \\ 2^m \displaystyle\prod_{i=1}^m t_{ii}^{\beta(i-1)+1}(d\mathbf{T}) & \text{if } \mathbf{S} = \mathbf{T}\mathbf{T}^*. \end{cases} \tag{2.11}$$

Proposition 2.3. (Spectral decomposition) *Let $\mathbf{S} \in \mathfrak{P}_m^\beta$. Then, the spectral decomposition can be written as $\mathbf{S} = \mathbf{W\Lambda W}^*$, where $\mathbf{W} \in \mathfrak{U}^\beta(m)$ and $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m) \in \mathfrak{D}_m^1$, with $\lambda_1 > \cdots > \lambda_m > 0$. Then*

- *parameter count: $\beta m(m-1)/2 + m = [\beta m(m+1)/2 - m - (\beta - 1)m] + [m]$ and*

$$(d\mathbf{S}) = 2^{-m}\pi^\varrho \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta (d\mathbf{\Lambda})(\mathbf{W}^* d\mathbf{W}), \tag{2.12}$$

where

$$\varrho = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

3 Integral Properties of Zonal Spherical Functions

In this section we review and derive several integral properties of zonal spherical functions for normed division algebras. However, let us first consider the following remarks and definitions in terms of Jack polynomials.

Remark 3.1. Note that Jack polynomials and hypergeometric functions with one or two matrix arguments are valid for $\beta > 0$ ([33]), but *in our case* β denotes the *real dimension* of $\mathfrak{F}$. Also, we use the parameter β indeed of α in the definition of the Jack polynomials and hypergeometric functions, with the equivalence shown in Table 1.

Then:

Let us characterise the *Jack symmetric function* $J_\kappa^{(\beta)}(\lambda_1,\dots,\lambda_m)$ of parameter β, see [40]. A decreasing sequence of nonnegative integers $\kappa = (k_1, k_2, \dots)$ with only finitely many nonzero terms is said to be a partition of $k = \sum k_i$. Let κ and $\tau = (t_1, t_2, \dots)$ be two partitions of k. We write $\tau \le \kappa$ if $\sum_{i=1}^t t_i \le \sum_{i=1}^t k_i$ for each t. The conjugate of κ is $\kappa' = (k'_1, k'_2, \dots)$ where $k'_i = \operatorname{card}\{j : k_j \ge i\}$. The length of κ is $l(k) = \max\{i : k_i \ne 0\} = k'_1$. If $l(\kappa) \le m$, it is often written that $\kappa = (k_1, k_2, \dots, k_m)$.

The *monomial symmetric function* $M_\kappa(\cdot)$ indexed by a partition κ can be regarded as a function of an arbitrary number of variables such that all but a finite number are equal to 0: if $\lambda_i = 0$ for $i > m \ge l(\kappa)$ then $M_\kappa(\lambda_1,\dots,\lambda_m) = \sum \lambda_1^{\delta_1} \cdots \lambda_m^{\delta_m}$, where the sum is over all distinct permutations $\{\delta_1,\dots,\delta_m\}$ of $\{k_1,\dots,k_m\}$, and if $l(\kappa) > m$ then $M_\kappa(\lambda_1,\dots,\lambda_m) = 0$. A symmetric function f is a linear combination of monomial symmetric functions. If f is a symmetric function then $f(\lambda_1,\dots,\lambda_m,0) = f(\lambda_1,\dots,\lambda_m)$. For each $m \ge 1$, $f(\lambda_1,\dots,\lambda_m)$ is a symmetric polynomial in m variables.

Then the *Jack symmetric function* $J_\kappa^{(\beta)}(\lambda_1,\dots,\lambda_m)$ with a parameter β, satisfies the following conditions:

$$J_\kappa^{(\beta)}(\lambda_1,\dots,\lambda_m) = \sum_{\tau\le\kappa} \nu_{\kappa,\tau}(\beta) M_\tau(\lambda_1,\dots,\lambda_m), \tag{3.1}$$

$$J_\kappa^{(\beta)}(1,\dots,1) = \left(\frac{2}{\beta}\right)^k \prod_{i=1}^m ((m-i+1)\beta/2)_{k_i}, \tag{3.2}$$

$$\mathcal{D}_2^\beta J_\kappa^{(\beta)}(\lambda_1,\dots,\lambda_m) = \sum_{i=1}^m k_i(k_i - 1 + \beta(m-i)) J_\kappa^{(\beta)}(\lambda_1,\dots,\lambda_m) \tag{3.3}$$

where

$$\mathcal{D}_2^\beta = \sum_{i=1}^{m} \lambda_i^2 \frac{\partial^2}{\partial \lambda_i^2} + \beta \sum_{i=1}^{m} \lambda_i^2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \frac{\partial}{\partial \lambda_i}.$$

Here, the constants $\nu_{\kappa,\tau}(\beta)$ do not dependent on $\lambda_i's$ but on κ and τ. Note that if $m < l(\kappa)$ then $J_\kappa^{(\beta)}(\lambda_1, \dots, \lambda_m) = 0$. The conditions include the case $\beta = 0$ and then $J_\kappa^{(0)}(\lambda_1, \dots, \lambda_m) = e_{\kappa'} \prod_{i=1}^{m} (m - i + 1)^{k_i}$, where $e_\kappa(\lambda_1, \dots, \lambda_m) = \prod_{i=1}^{l(\kappa)} e_{k_i}(\lambda_1, \dots, \lambda_m)$ are the elementary symmetric functions indexed by partitions κ, if $m \geq l(\kappa)$ then $e_r(\lambda_1, \dots, \lambda_m) = \sum_{i_1 < i_2 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}$, and if $m < l(\kappa)$ then $e_r(\lambda_1, \dots, \lambda_m) = 0$, see [40].

Now, from [33], the Jack functions

$$J_\kappa^{(\beta)}(\mathbf{X}) = J_\kappa^{(\beta)}(\lambda_1, \dots, \lambda_m),$$

where $\lambda_1, \dots, \lambda_m$ are the eigenvalues of the matrix $\mathbf{X} \in \mathfrak{S}_m^\beta$, can be normalised in such a way that

$$\sum_\kappa C_\kappa^\beta(\mathbf{X}) = (\mathrm{tr}(\mathbf{X}))^k, \tag{3.4}$$

or equivalently, such that

$$\sum_{k=1}^{\infty} \sum_\kappa \frac{C_\kappa^\beta(\mathbf{X})}{k!} = \mathrm{etr}\{\mathbf{X}\}, \tag{3.5}$$

where $C_\kappa^\beta(\mathbf{X})$ denotes the *Jack polynomials* (for simplicity, we have replaced (β) by β as the superindex for the Jack polynomials). These are related to the Jack functions by

$$C_\kappa^\beta(\mathbf{X}) = \frac{2^k k!}{\beta^k \nu_\kappa} J_\kappa^{(\beta)}(\mathbf{X}), \tag{3.6}$$

where

$$\nu_\kappa = \prod_{(i,j) \in \kappa} h_*^\kappa(i,j) h_\kappa^*(i,j),$$

and $h_*^\kappa(i,j) = k_j - i + 2(k_i - j + 1)/\beta$ and $h_\kappa^*(i,j) = k_j - i + 1 + 2(k_i - j)/\beta$ are the upper and lower hook lengths at $(i,j) \in \kappa$, respectively.

From this point we return to the particular case of zonal spherical functions. Then, observe that for $\mathbf{X} = \mathbf{S}^*\mathbf{S}$ and $\mathbf{Y} = \mathbf{W}^*\mathbf{W}$ we have

$$C_\kappa^\beta(\mathbf{W}\mathbf{X}\mathbf{W}^*) = C_\kappa^\beta(\mathbf{S}\mathbf{Y}\mathbf{S}^*). \tag{3.7}$$

In particular for $\mathbf{A}^{1/2}$ such that $\left(\mathbf{A}^{1/2}\right)^2 = \mathbf{A}$

$$C_\kappa^\beta(\mathbf{Y}^{1/2}\mathbf{X}\mathbf{Y}^{1/2}) = C_\kappa^\beta(\mathbf{X}^{1/2}\mathbf{Y}\mathbf{X}^{1/2}). \tag{3.8}$$

Therefore, given that $\mathbf{XY}$, $\mathbf{YX}$, $\mathbf{Y}^{1/2}\mathbf{X}\mathbf{Y}^{1/2}$ and $\mathbf{X}^{1/2}\mathbf{Y}\mathbf{X}^{1/2}$ all have the same eigenvalues, for convenience of notation rather than strict adherence to rigor, we write $C_\kappa^\beta(\mathbf{XY})$ or $C_\kappa^\beta(\mathbf{YX})$ rather than $C_\kappa^\beta(\mathbf{Y}^{1/2}\mathbf{X}\mathbf{Y}^{1/2})$, even though $\mathbf{XY}$ or $\mathbf{YX}$ need not lie in $\mathfrak{S}_m^\beta$. Note that

$$C_\kappa^\beta(\mathbf{Z}^{1/2}\mathbf{X}\mathbf{Z}^{1/2}\mathbf{Y}) = C_\kappa^\beta(\mathbf{X}\mathbf{Z}^{1/2}\mathbf{Y}\mathbf{Z}^{1/2}), \tag{3.9}$$

for all $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^\beta$ and $\mathbf{Z} \in \mathfrak{P}_m^\beta$. From [23, Equation 4.8(2) and Definition 5.3] we have

$$C_\kappa^\beta(\mathbf{X}) = C_\kappa^\beta(\mathbf{I}) \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} q_\kappa(\mathbf{H}^*\mathbf{X}\mathbf{H})(d\mathbf{H}) \tag{3.10}$$

for all $\mathbf{X} \in \mathfrak{S}_m^\beta$; where $(d\mathbf{H})$ is the normalised Haar measure on $\mathfrak{U}^\beta(m)$. Finally, for c constant we have that $C_\kappa^\beta(c\mathbf{X}) = c^k C_\kappa^\beta(\mathbf{X})$.

Some basic integral properties are cited below. For this purpose, we utilise the complexification $\mathfrak{S}_m^{\beta,\mathfrak{C}} = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ of $\mathfrak{S}_m^\beta$. That is, $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consist of all matrices $\mathbf{Z} \in (\mathfrak{F}^\mathfrak{C})^{m\times m}$ of the form $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^\beta$. We refer to $\mathbf{X} = \mathrm{Re}(\mathbf{Z})$ and $\mathbf{Y} = \mathrm{Im}(\mathbf{Z})$ as the *real and imaginary parts* of $\mathbf{Z}$, respectively. The *generalised right half-plane* $\mathbf{\Phi} = \mathfrak{P}_m^\beta + i\mathfrak{S}_m^\beta$ in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ such that $\mathrm{Re}(\mathbf{Z}) \in \mathfrak{P}_m^\beta$, see [23, p. 801].

For any $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^\beta$,

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} C_\kappa^\beta(\mathbf{X}\mathbf{H}^*\mathbf{Y}\mathbf{H})(d\mathbf{H}) = \frac{C_\kappa^\beta(\mathbf{X})C_\kappa^\beta(\mathbf{Y})}{C_\kappa^\beta(\mathbf{I})}. \tag{3.11}$$

For all $\mathbf{R} \in \mathfrak{S}_m^\beta$, $\mathbf{Z} \in \mathbf{\Phi}$ and $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$,

$$\begin{aligned}\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} &\mathrm{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\kappa^\beta(\mathbf{XR})(d\mathbf{X}) \\ &= \Gamma_m^\beta[a,\kappa]|\mathbf{Z}|^{-a} C_\kappa^\beta(\mathbf{R}\mathbf{Z}^{-1}) \\ &= [a]_\kappa^\beta \Gamma_m^\beta[a]|\mathbf{Z}|^{-a} C_\kappa^\beta(\mathbf{R}\mathbf{Z}^{-1}). \end{aligned} \tag{3.12}$$

Remark 3.2. In general, the result (3.12) has been established under the condition, $\mathrm{Re}(a) > (m-1)\beta/2$, see [6], [37], [39] and [34], but in reality the correct condition is $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$. This fact is immediate, observing that $[a]_\kappa^\beta \Gamma_m^\beta[a] = \Gamma_m^\beta[a,\kappa]$ and the different expressions for $\Gamma_m^\beta[a,\kappa]$ in (2.4).

Let $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$ and $\mathrm{Re}(b) > (m-1)\beta/2$. Then

$$\begin{aligned}\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} &|\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{XR})(d\mathbf{X})\\ &= \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,\kappa]}C_\kappa^\beta(\mathbf{R})\\ &= \frac{[a]_\kappa^\beta\mathcal{B}_m^\beta[a,b]}{[a+b]_m^\beta}C_\kappa^\beta(\mathbf{R}), \quad (3.13)\end{aligned}$$

for all $\mathbf{R} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$; see [23, Theorems 5.5 and 5.9 and Corollary 5.10] for real, complex and quaternion cases.

Remark 3.3. Observe that result (3.13) was established under the conditions $\mathrm{Re}(a) > (m-1)\beta/2$ and $\mathrm{Re}(b) > (m-1)\beta/2$, see [6], [37], [39] and [34], but the correct conditions are in fact $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$ and $\mathrm{Re}(b) > (m-1)\beta/2$. This fact is verified by observing that $[a]_\kappa^\beta\Gamma_m^\beta[a] = \Gamma_m^\beta[a,\kappa]$ and the different expressions for $\Gamma_m^\beta[a,\kappa]$ in (2.4).

We now extend several integral properties of zonal polynomials in the real and complex cases to normed division algebras. Our first result is a generalisation of one studied by [42] for real case, see also [3]. From this result, we can obtain diverse particular integral properties of zonal spherical functions.

Theorem 3.1. *Let $\mathbf{Z} \in \mathbf{\Phi}$ and $\mathbf{U} \in \mathfrak{S}_m^\beta$. Assume*

$$\gamma = \int_{z\in\mathfrak{P}_1^\beta} f(z)z^{am-k-1}dz < \infty.$$

Then

$$\begin{aligned}\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} &f(\mathrm{tr}\,\mathbf{XZ})|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta\left(\mathbf{X}^{-1}\mathbf{U}\right)(d\mathbf{X})\\ &= \frac{\Gamma_m^\beta[a,-\kappa]}{\Gamma[am-k]}|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{UZ})\cdot\gamma, \qquad (3.14)\end{aligned}$$

for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$, *and*

$$\begin{aligned}\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} &f(\mathrm{tr}\,\mathbf{XZ})|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta\left(\mathbf{XU}\right)(d\mathbf{X})\\ &= \frac{\Gamma_m^\beta[a,\kappa]}{\Gamma[am+k]}|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{UZ}^{-1})\cdot\vartheta, \qquad (3.15)\end{aligned}$$

where $\vartheta = \int_{z\in\mathfrak{P}_1^\beta} f(z)z^{am+k-1}dz < \infty$, $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$ *and* $\kappa = (k_1,\ldots,k_m)$ *and* $k = k_1+\cdots+k_m$.

Proof. Denote the left side of (3.14) by $I(\mathbf{U},\mathbf{Z})$. By (3.10) and interchange of order on integration

$$\begin{aligned}
I(\mathbf{I},\mathbf{I}) &= \int_{\mathbf{X}\in\mathfrak{P}_m^\beta} f(\operatorname{tr}\mathbf{X})|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\kappa^\beta\left(\mathbf{X}^{-1}\right)(d\mathbf{X}) \\
&= C_\kappa^\beta(\mathbf{I}) \int_{\mathbf{X}\in\mathfrak{P}_m^\beta} f(\operatorname{tr}\mathbf{X})|\mathbf{X}|^{a-(m-1)\beta/2-1} \\
&\qquad \times \left(\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} q_\kappa\left(\mathbf{H}^*\mathbf{X}^{-1}\mathbf{H}\right)(d\mathbf{H})\right)(d\mathbf{X}) \\
&= C_\kappa^\beta(\mathbf{I}) \int_{\mathbf{X}\in\mathfrak{P}_m^\beta} f(\operatorname{tr}\mathbf{X})|\mathbf{X}|^{a-(m-1)\beta/2-1} q_\kappa\left(\mathbf{X}^{-1}\right)(d\mathbf{X}).
\end{aligned}$$

Let $\mathbf{X}=\mathbf{T}\mathbf{T}^*$, from Proposition 2.2

$$(d\mathbf{X}) = 2^m \prod_{i=1}^m t_{ii}^{\beta(i-1)+1}(d\mathbf{T}).$$

Then

$$\mathcal{I}(\mathbf{I},\mathbf{I}) = 2^m C_\kappa^\beta(\mathbf{I}) \underset{\substack{0<t_{ii}<\infty \\ -\infty<t_{ij}<\infty}}{\int\cdots\int} f\left(\sum_{i\le j}^m t_{ij}^2\right)\prod_{i=1}^m (t_{ii})^{2(a-k_i-(m-i)\beta/2)-1}(d\mathbf{T}).$$

Applying [18, Lemma 2.4.3, p. 51] we obtain

$$\begin{aligned}
\mathcal{I}(\mathbf{I},\mathbf{I}) &= C_\kappa^\beta(\mathbf{I}) \frac{\pi^{m(m-1)\beta/4}\displaystyle\prod_{i=1}^m \Gamma[a-k_i-(m-i)\beta/2]}{\Gamma[am-k]}\cdot\gamma \\
&= C_\kappa^\beta(\mathbf{I})\frac{\Gamma_m^\beta[a,-\kappa]}{\Gamma[am-k]}\cdot\gamma,
\end{aligned}$$

with $\gamma=\int_{z\in\mathfrak{P}_1^\beta} f(z)z^{am-k-1}dz<\infty$.

Next, since the function $\mathcal{I}(\mathbf{U},\mathbf{I})$ is invariant under $\mathfrak{U}^\beta(m)$ and in $\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$, there exists a constant d such that $\mathcal{I}(\mathbf{U},\mathbf{I}) = dC_\kappa^\beta(\mathbf{U})$, $\mathbf{U}\in\mathfrak{S}_m^\beta$. It is obvious that $d=\mathcal{I}(\mathbf{I},\mathbf{I})/C_\kappa^\beta(\mathbf{I})$, then

$$\mathcal{I}(\mathbf{U},\mathbf{I}) = \frac{\Gamma_m^\beta[a,-\kappa]}{\Gamma[am-k]}C_\kappa^\beta(\mathbf{U})\cdot\gamma.$$

Now, let $\mathbf{Z}\in\mathfrak{P}_m^\beta$ and make the change of variable $\mathbf{X}\to\mathbf{Z}^{-1/2}\mathbf{X}\mathbf{Z}^{-1/2}$ in the integral defining $\mathcal{I}(\mathbf{U},\mathbf{Z})$. Then by (3.9)

$$\mathcal{I}(\mathbf{U},\mathbf{Z}) = |\mathbf{Z}|^{-a}\mathcal{I}(\mathbf{Z}^{1/2}\mathbf{U}\mathbf{Z}^{1/2},\mathbf{I}),$$

and hence

$$\mathcal{I}(\mathbf{U},\mathbf{Z}) = \frac{\Gamma_m^\beta[a,-\kappa]}{\Gamma[am-k]}|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{Z}^{1/2}\mathbf{U}\mathbf{Z}^{1/2})\cdot\gamma.$$

Therefore, for $\mathbf{Z}\in\mathfrak{P}_m^\beta$ and $\mathbf{U}\in\mathfrak{S}_m^\beta$

$$\mathcal{I}(\mathbf{U},\mathbf{Z}) = \frac{\Gamma_m^\beta[a,-\kappa]}{\Gamma[am-k]}|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{U}\mathbf{Z})\cdot\gamma.$$

The result in (3.14) now follows by analytic continuation in $\mathbf{Z}$ from $\mathfrak{P}_m^\beta$ to $\mathbf{\Phi}=\mathfrak{P}_m^\beta+i\mathfrak{S}_m^\beta$. The result in (3.15) is obtained in a similar way. □

Now, by definition if $\kappa=0$ then $[a]_\kappa^\beta=1$ and $C_\kappa^\beta(\mathbf{X})=1$ from where:

Corollary 3.1. *Let* $\mathbf{Z}\in\mathbf{\Phi}\in\mathfrak{S}_m^\beta$*. Assume* $\gamma=\int_{z\in\mathfrak{P}_1^\beta}f(z)z^{am-1}dz<\infty$*. Then*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} f(\operatorname{tr}\mathbf{X}\mathbf{Z})|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X}) = \frac{\Gamma_m^\beta[a]}{\Gamma[am]}|\mathbf{Z}|^{-a}\cdot\gamma, \tag{3.16}$$

for $\operatorname{Re}(a)>(m-1)\beta/2$.

If we take $f(y)=\exp\{-y\}$ in Corollary 3.1 we obtain

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta}\operatorname{etr}\{-\mathbf{X}\mathbf{Z}\}|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X}) = \Gamma_m^\beta[a]|\mathbf{Z}|^{-a}, \tag{3.17}$$

and if $\mathbf{Z}=\mathbf{I}$ we obtain the multivariate gamma function for the space $\mathfrak{S}_m^\beta$.

Other particular results of Theorem 3.1 are summarised below:

Defining $f(\operatorname{tr}\mathbf{X}\mathbf{Z}) = \operatorname{etr}\{-\mathbf{X}\mathbf{Z}\}(\operatorname{tr}\mathbf{X}\mathbf{Z})^j$, or equivalently $f(y)=\exp\{-y\}y^j$, $j\in\Re$, it is obtained:

Corollary 3.2. *Let* $\mathbf{Z}\in\mathbf{\Phi}$ *and* $\mathbf{U}\in\mathfrak{S}_m^\beta$ *and* $j\in\Re$*, such that* $\operatorname{Re}(ma+j-k)>0$*, then*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta}\operatorname{etr}\{-\mathbf{X}\mathbf{Z}\}(\operatorname{tr}\mathbf{X}\mathbf{Z})^j|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta\left(\mathbf{X}^{-1}\mathbf{U}\right)(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,-\kappa]\Gamma[ma+j-k]}{\Gamma[ma-k]}|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{U}\mathbf{Z}), \tag{3.18}$$

for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$. *And if* j *is such that* $\mathrm{Re}(ma+j+k) > 0$, *then*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{XZ}\}(\operatorname{tr}\mathbf{XZ})^j|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{XU})(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,\kappa]\Gamma[ma+j+k]}{\Gamma[ma+k]}|\mathbf{Z}|^{-a}C_\kappa^\beta\left(\mathbf{UZ}^{-1}\right), \quad (3.19)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$.

Now if $j = 0$ is defined in Corollary 3.3 we have:

Corollary 3.3. *Let* $\mathbf{Z} \in \mathbf{\Phi}$ *and* $\mathbf{U} \in \mathfrak{S}_m^\beta$.

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta\left(\mathbf{X}^{-1}\mathbf{U}\right)(d\mathbf{X})$$
$$= \Gamma_m^\beta[a,-\kappa]|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{UZ}), \quad (3.20)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$. *And*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{XU})(d\mathbf{X})$$
$$= \Gamma_m^\beta[a,\kappa]|\mathbf{Z}|^{-a}C_\kappa^\beta\left(\mathbf{UZ}^{-1}\right), \quad (3.21)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$.

Corollary 3.4. *Let* $\mathbf{Z} \in \mathbf{\Phi}$ *and* $\mathbf{U} \in \mathfrak{S}_m^\beta$ *and* $\eta > 0$ *then*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} (1+2\eta^{-1}\operatorname{tr}\mathbf{XZ})^{-\beta(am+\eta)}|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta\left(\mathbf{X}^{-1}\mathbf{U}\right)(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,-\kappa]\Gamma[(\beta-1)am+\beta\eta+k]}{(2\eta^{-1})^{am-k}\Gamma[\beta(ma+\eta)]}|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{UZ}), \quad (3.22)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$. *And*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} (1+2\eta^{-1}\operatorname{tr}\mathbf{XZ})^{-\beta(am+\eta)}|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{XU})(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,\kappa]\Gamma[(\beta-1)am+\beta\eta-k]}{(2\eta^{-1})^{am+k}\Gamma[\beta(ma+\eta)]}|\mathbf{Z}|^{-a}C_\kappa^\beta(\mathbf{UZ}^{-1}), \quad (3.23)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$.

Proof. The desired result is obtained by taking $f(y) = (1+2\eta^{-1}y)^{-\beta(am+\eta)}$ in Theorem 3.1. □

Many other interesting particular cases of Theorem 3.1 can be found, for example by defining $f(\operatorname{tr}\mathbf{XZ})$ as the kernel of matrix variate generalised Wishart distributions, see [18] and [26].

Important analogues of the beta function integral are given in the following theorems. Theorem 3.2 is discussed by [31] in the real case.

Theorem 3.2. *If* $\mathbf{R} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, *then*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\kappa^\beta(\mathbf{R}\mathbf{X}^{-1})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,-\kappa]\Gamma_m^\beta[b,\kappa]}{\Gamma_m^\beta[a+b]}C_\kappa^\beta(\mathbf{R}), \tag{3.24}$$

for $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$ *and* $\operatorname{Re}(b) > (m-1)\beta/2 - k_m$. *And*

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\kappa^\beta(\mathbf{R}\mathbf{X})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,-\kappa]}{\Gamma_m^\beta[a+b]}C_\kappa^\beta(\mathbf{R}), \tag{3.25}$$

for $\operatorname{Re}(a) > (m-1)\beta/2 - k_m$ *and* $\operatorname{Re}(b) > (m-1)\beta/2 + k_1$.

Proof. By Corollary 3.3, we have for any $\mathbf{Z} \in \mathbf{\Phi}$

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta\left(\mathbf{X}^{-1}\mathbf{R}\right)|\mathbf{Z}|^a(d\mathbf{X}) = \Gamma_m^\beta[a,-\kappa]C_\kappa^\beta(\mathbf{RZ}). \tag{3.26}$$

Multiplying both sides of (3.26) by $\operatorname{etr}\{-\mathbf{Z}\}|\mathbf{Z}|^{b-(m-1)\beta/2-1}$ and integrating with respect to $\mathbf{Z}$ we have

$$\begin{aligned}\int_{\mathbf{X}\in\mathfrak{P}_m^\beta}&\left(\int_{\mathbf{Z}\in\mathfrak{P}_m^\beta}\operatorname{etr}\{-(\mathbf{I}+\mathbf{X})\mathbf{Z}\}|\mathbf{Z}|^{a+b-(m-1)\beta/2-1}(d\mathbf{Z})\right)\\ &\times|\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta\left(\mathbf{X}^{-1}\mathbf{R}\right)(d\mathbf{X})\\ &=\Gamma_m^\beta[a,-\kappa]\int_{\mathbf{Z}\in\mathfrak{P}_m^\beta}\operatorname{etr}\{-\mathbf{Z}\}|\mathbf{Z}|^{b-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{RZ})(d\mathbf{Z}).\end{aligned} \tag{3.27}$$

The desired result in (3.24) is obtained by using (3.17) and (3.21) in the left and right sides of (3.27), respectively. The result in (3.25) is obtained similarly. □

Corollary 3.5. *If* $\mathbf{R} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, *then*

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{R}\mathbf{X}^{-1})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,-\kappa]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,-\kappa]}C_\kappa^\beta(\mathbf{R}), \qquad (3.28)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$ *and* $\mathrm{Re}(b) > (m-1)\beta/2$.

Proof. It is obtained in a similar way to that given for (3.13), see [23]. □

Now, taking $b = (m-1)\beta/2 + 1 > (m-1)\beta/2$, from (3.13) and Corollary 3.5 we have the following result.

Corollary 3.6. *If* $\mathbf{R} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, *then*

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{R}\mathbf{X}^{-1})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,-\kappa]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,-\kappa]}C_\kappa^\beta(\mathbf{R}), \qquad (3.29)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$. *And*

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{X}\mathbf{R})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,\kappa]}C_\kappa^\beta(\mathbf{R}), \qquad (3.30)$$

for $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$.

Similarly, taking $a = (m-1)\beta/2 + 1 > (m-1)\beta/2 - k_m$ in (3.13), we have the following result.

Corollary 3.7. *If* $\mathbf{R} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, *then*

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\kappa^\beta(\mathbf{X}\mathbf{R})(d\mathbf{X}) = \frac{\Gamma_m^\beta[(m-1)\beta/2+1,\kappa]\Gamma_m^\beta[b]}{\Gamma_m^\beta[(m-1)\beta/2+1+b,\kappa]}C_\kappa^\beta(\mathbf{R}), \qquad (3.31)$$

for $\mathrm{Re}(b) > (m-1)\beta/2$.

4 Hypergeometric Functions

In this section, we study diverse integral properties of hypergeometric functions for normed division algebras. First, let us consider the following definition.

Fix complex numbers $a_1, \dots, a_p$ and $b_1, \dots, b_q$, and for all $1 \le i \le q$ and $1 \le j \le m$ do not allow $-b_i + (j-1)\beta/2$ to be a nonnegative integer. Then the *hypergeometric function with one matrix argument* ${}_pF_q^\beta$ is defined to be the real-analytic function on $\mathfrak{S}_m^\beta$ given by the series

$$ {}_pF_q^\beta(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{X}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_\kappa^\beta \cdots [a_p]_\kappa^\beta}{[b_1]_\kappa^\beta \cdots [b_q]_\kappa^\beta} \frac{C_\kappa^\beta(\mathbf{X})}{k!}. \tag{4.1} $$

Some known properties are, see [23, Section 6, pp. 803-810]:
Convergence hypergeometric functions.

1. If $p \le q$ then the hypergeometric series (4.1) converges absolutely for all $\mathbf{X} \in \mathfrak{S}_m^\beta$.
2. If $p = q+1$ then the series (4.1) converges absolutely for $||\mathbf{X}|| = \max\{|\lambda_i| : i = 1, \dots, m\} < 1$, and diverges for $||\mathbf{X}|| > 1$, where $\lambda_1, \dots \lambda_m$ are the i-th eigenvalues of $\mathbf{X} \in \mathfrak{S}_m^\beta$.
3. If $p > q$ then the series (4.1) diverges unless it terminates.

For all $\mathbf{X} \in \mathfrak{S}_m^\beta$; indeed, for all $\mathbf{X} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$. This is characteristic of the general situation when $p \le q$.

$$ {}_0F_0^\beta(\mathbf{X}) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa^\beta(\mathbf{X})}{k!} = \sum_{k=0}^{\infty} \frac{(\operatorname{tr} \mathbf{X})^k}{k!} = \operatorname{etr}\{\mathbf{X}\}, \tag{4.2} $$

If $\operatorname{Re}(a) > (m-1)\beta/2$, and $||\mathbf{X}|| < 1$,

$$ \begin{aligned} {}_1F_0^\beta(a; \mathbf{X}) &= \frac{1}{\Gamma_m^\beta[a]} \int_{\mathbf{Y} \in \mathfrak{P}_m^\beta} \operatorname{etr}\{-(\mathbf{I} - \mathbf{X})\mathbf{Y}\} |\mathbf{Y}|^{a-(m-1)\beta/2-1} (d\mathbf{Y}) \\ &= |\mathbf{I} - \mathbf{X}|^{-a} \end{aligned} \tag{4.3} $$

gives the full analytic continuation of ${}_1F_0^\beta(a; \cdot)$ to any simply-connected domain in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$. The right side is determined by the principal branch of the argument. The fact that the hypergeometric series ${}_1F_0^\beta$ has $\{\mathbf{X} \in \mathfrak{S}_m^\beta : ||\mathbf{X}|| < 1\}$ as its domain of convergence is characteristic of ${}_{p+1}F_p^\beta$ for all $p \ge 0$.

Let $\mathrm{Re}(c) > \mathrm{Re}(a) + (m-1)\beta/2 > (m-1)\beta$ and $||\mathbf{X}|| < 1$. Then

$$\begin{aligned}
{}_{p+1}F_{q+1}^{\beta}&(a_1,\dots,a_p,a;b_1,\dots,b_q,c;\mathbf{X}) \\
&= \frac{1}{\mathcal{B}_m^{\beta}[a,c-a]}\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} {}_pF_q^{\beta}(a_1\cdots a_p;b_1\cdots b_q;\mathbf{XY}) \\
&\quad \times|\mathbf{Y}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{Y}|^{c-a-(m-1)\beta/2-1}(d\mathbf{Y}), \qquad (4.4)
\end{aligned}$$

for $p = q+1$. In particular, for $p = 1$, we have the Euler formula

$$\begin{aligned}
{}_2F_1^{\beta}(a_1,a;c;\mathbf{X}) &= \frac{1}{\mathcal{B}_m^{\beta}[a,c-a]} \\
&\quad \times\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} {}_1F_0^{\beta}(a_1;\mathbf{XY})|\mathbf{Y}|^{a-(m-1)\beta/2-1} \\
&\quad \times|\mathbf{I}-\mathbf{Y}|^{c-a-(m-1)\beta/2-1}(d\mathbf{Y}). \qquad (4.5)
\end{aligned}$$

for arbitrary a_1, $\mathrm{Re}(c) > \mathrm{Re}(a) + (m-1)\beta/2 > (m-1)\beta$ and $||\mathbf{X}|| < 1$.

Remark 4.1. Observe that, by expanding in (4.4) the hypergeometric function in terms of Jack polynomials, it can be expressed as

$$\begin{aligned}
&{}_{p+1}F_{q+1}^{\beta}(a_1,\dots,a_p,a;b_1,\dots,b_q,c;\mathbf{X}) \\
&= \frac{1}{\mathcal{B}_m^{\beta}[a,c-a]}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa^\beta\cdots[a_p]_\kappa^\beta}{[b_1]_\kappa^\beta\cdots[b_q]_\kappa^\beta}\int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}_m} C_\kappa^\beta(\mathbf{XY}) \\
&\quad \times|\mathbf{Y}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{Y}|^{c-a-(m-1)\beta/2-1}(d\mathbf{Y}), \qquad (4.6)
\end{aligned}$$

which is a consequence of (3.13). And then, for all ordered partition $\kappa = (k_1,\dots,k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$ of $k = \sum_{i=1}^m k_i$, $k = 0,1,\dots,$ the condition over a and c must be $\mathrm{Re}(c) > \mathrm{Re}(a)+(m-1)\beta/2 > (m-1)\beta-k_m$. However observe that, in particular, if $\mathrm{Re}(a) > (m-1)\beta/2-k_m$ then $\mathrm{Re}(a) > (m-1)\beta/2$ because $k_m = 0,1,\dots$ Therefore although some of the integral in (4.4) are valid for other values of a, specifically for $\mathrm{Re}(a) > (m-1)\beta/2-k_m$, all exist if $\mathrm{Re}(a) > (m-1)\beta/2$. Hence the condition over a and c which guarantee the existence of (4.4) is $\mathrm{Re}(c) > \mathrm{Re}(a) + (m-1)\beta/2 > (m-1)\beta$. Similar reflection applies to (4.5).

Laplace transform of hypergeometric functions. Assume $p \leq q$, $\mathrm{Re}(a) >$

$(m-1)\beta/2$ and $\mathbf{U} \in \mathfrak{S}_m^\beta$. Then

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{XZ}\}{}_pF_q^\beta(a_1\cdots a_p; b_1\cdots b_q; \mathbf{XU})|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X})$$
$$= |\mathbf{Z}|^{-a}\Gamma_m^\beta[a]\ {}_{p+1}F_q^\beta(a_1\cdots a_p, a; b_1\cdots b_q; \mathbf{UZ}^{-1}). \quad (4.7)$$

When $p < q$, the integral in (4.7) converges absolutely for all $\mathbf{Z} \in \mathbf{\Phi}$. When $p = q$, the integral converges absolutely for all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, such that $||(\mathrm{Re}(\mathbf{Z}))^{-1}|| < 1$.

Similarly, the *hypergeometric function of two matrix arguments* ${}_pF_q^{(m),\beta}$ is defined to be the real-analytic function on $\mathfrak{S}_m^\beta$ given by the series

$${}_pF_q^{(m),\beta}(a_1,\dots,a_p; b_1,\dots,b_q; \mathbf{X},\mathbf{Y}) = \sum_{k=0}^\infty \sum_\kappa \frac{[a_1]_\kappa^\beta\cdots[a_p]_\kappa^\beta}{[b_1]_\kappa^\beta\cdots[b_q]_\kappa^\beta}\ \frac{C_\kappa^\beta(\mathbf{X})C_\kappa^\beta(\mathbf{Y})}{k!\ C_\kappa^\beta(\mathbf{I})}. \quad (4.8)$$

Some basic properties of (4.8) are shown below, see [24].

Convergence hypergeometric functions with two matrix arguments.

1. If $p \le q$ then the hypergeometric series (4.8) converges absolutely for all $\mathbf{X}$ and $\mathbf{Y} \in \mathfrak{S}_m^\beta$.
2. If $p = q + 1$ then the series (4.8) converges absolutely for $||\mathbf{X}|| \cdot ||\mathbf{Y}|| < 1$, and diverges for $||\mathbf{X}|| \cdot ||\mathbf{Y}|| > 1$.

Also

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} {}_pF_q^\beta(a_1,\dots,a_p; b_1,\dots,b_q; \mathbf{XHYH}^*)(d\mathbf{H}) \quad (4.9)$$
$$= {}_pF_q^{(m),\beta}(a_1,\dots,a_p; b_1,\dots,b_q; \mathbf{X},\mathbf{Y}) \quad (4.10)$$

In particular

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} {}_0F_0^\beta(\mathbf{XHYH}^*)(d\mathbf{H}) = \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \operatorname{etr}\{\mathbf{XHYH}^*\}(d\mathbf{H}) \quad (4.11)$$
$$= {}_0F_0^{(m),\beta}(\mathbf{X},\mathbf{Y}). \quad (4.12)$$

Laplace transform of hypergeometric functions with two matrix arguments.

Assume $p \le q$, $\mathrm{Re}(a) > (m-1)\beta/2$ and $\mathbf{U} \in \mathfrak{S}_m^\beta$. Then

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{XZ}\}{}_pF_q^{(m),\beta}(a_1\cdots a_p; b_1\cdots b_q; \mathbf{XU},\mathbf{Y})|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X})$$
$$= |\mathbf{Z}|^{-a}\Gamma_m^\beta[a]\ {}_{p+1}F_q^{(m),\beta}(a_1\cdots a_p, a; b_1\cdots b_q; \mathbf{UZ}^{-1},\mathbf{Y}). \quad (4.13)$$

When $p < q$, the integral in (4.13) converges absolutely for all $\mathbf{Z} \in \mathbf{\Phi}$ and $\mathbf{Y} \in \mathfrak{S}_m^\beta$. When $p = q$, the integral converges absolutely for all $\mathbf{Z}$ and $\mathbf{Y} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, such that $||(\mathrm{Re}(\mathbf{Z}))^{-1}|| \cdot ||\mathbf{Y}|| < 1$.

We now propose further integral properties of hypergeometric functions for normed division algebras. The first result is the inverse Laplace transformation. In the real case, this result was obtained by [27], [6], [29] and [37, p. 261], and in the complex case by [35, p. 370]. Let us first consider the following extension of similar results discussed in [6], see also [37, p. 253].

Proposition 4.1. *Assume that* $\mathbf{Z} = \mathbf{Z}_0 + i\mathbf{Y}$ *and* $\mathbf{X} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, $\mathbf{Y}$ *and* $\mathbf{U} \in \mathfrak{S}_m^\beta$ *and* $\mathrm{Re}(a) > a_0$. *Then*

$$\frac{2^{m(m-1)\beta/2}}{(2\pi i)^{m(m-1)\beta/2+m}} \int_{\mathrm{Re}(\mathbf{Z})=\mathbf{Z}_0\in\mathfrak{P}_m^\beta} \mathrm{etr}\{\mathbf{XZ}\}|\mathbf{Z}|^{-a} C_\kappa^\beta(\mathbf{UZ}^{-1})(d\mathbf{Z})$$
$$= \frac{1}{\Gamma_m^\beta[a,\kappa]}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\kappa^\beta(\mathbf{XU}). \quad (4.14)$$

Theorem 4.1. *Assume that* $\mathbf{Z} = \mathbf{Z}_0 + i\mathbf{V}$ *and* $\mathbf{X} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, $\mathbf{V}$ *and* $\mathbf{U} \in \mathfrak{S}_m^\beta$ *and* $\mathrm{Re}(b) > b_0$. *Then*

$$\frac{\Gamma_m^\beta[b]2^{m(m-1)\beta/2}}{(2\pi i)^{m(m-1)\beta/2+m}} \int_{\mathrm{Re}(\mathbf{Z})=\mathbf{Z}_0\in\mathfrak{P}_m^\beta} \mathrm{etr}\{\mathbf{XZ}\}|\mathbf{Z}|^{-b}$$
$$\times {}_pF_q^\beta(a_1,\dots,a_p;b_1,\dots,b_q;\mathbf{UZ}^{-1})(d\mathbf{Z})$$
$$= |\mathbf{X}|^{b-(m-1)\beta/2-1} {}_pF_{q+1}^\beta(a_1,\dots,a_p;b_1,\dots,b_q,b;\mathbf{XU}), \quad (4.15)$$

and if $\mathbf{Y} \in \mathfrak{S}_m^\beta$

$$\frac{\Gamma_m^\beta[b]2^{m(m-1)\beta/2}}{(2\pi i)^{m(m-1)\beta/2+m}} \int_{\mathrm{Re}(\mathbf{Z})=\mathbf{Z}_0\in\mathfrak{P}_m^\beta} \mathrm{etr}\{\mathbf{XZ}\}|\mathbf{Z}|^{-b}$$
$$\times \ {}_pF_q^{(m),\beta}(a_1,\dots,a_p;b_1,\dots,b_q;\mathbf{UZ}^{-1},\mathbf{Y})(d\mathbf{Z})$$
$$= |\mathbf{X}|^{b-(m-1)\beta/2-1} {}_pF_{q+1}^{(m),\beta}(a_1,\dots,a_p;b_1,\dots,b_q,b;\mathbf{XU},\mathbf{Y}), \quad (4.16)$$

where $\mathbf{Z}_0 \in \mathfrak{P}_m^\beta$.

Proof. Proof of both (4.15) and (4.16) follows by expanding the ${}_pF_q^\beta$ and ${}_pF_q^{(m),\beta}$ functions in the integrands and integrating term by term using (4.14). $\square$

Theorem 4.2. *The ${}_1F_1^\beta$ function has the integral representation*

$$\begin{aligned} {}_1F_1^\beta(a;c;\mathbf{X}) &= \frac{1}{\mathcal{B}_m^\beta[a,c-a]} \int_{\mathbf{0}<\mathbf{Y}<\mathbf{I}} \operatorname{etr}\{\mathbf{XY}\}|\mathbf{Y}|^{a-(m-1)\beta/2-1} \\ &\times \ |\mathbf{I}-\mathbf{Y}|^{c-a-(m-1)\beta/2-1}(d\mathbf{Y}), \end{aligned} \tag{4.17}$$

valid for $\operatorname{Re}(c) > \operatorname{Re}(a) + (m-1)\beta/2 > (m-1)\beta$ *and all* $\mathbf{X} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$.

Proof. The desired result is obtained by expanding $\operatorname{etr}\{\mathbf{XY}\}$ using (3.5) and integrating term by term using (3.13). □

The generalised Kummer and Euler relations are given in the following result.

Theorem 4.3.

$${}_1F_1^\beta(a;c;\mathbf{X}) = \operatorname{etr}\{\mathbf{X}\}{}_1F_1^\beta(c-a,c;-\mathbf{X}), \tag{4.18}$$

for $\mathbf{X} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$. *And*

$$\begin{aligned} {}_2F_1^\beta(a,b;c;\mathbf{X}) &= |\mathbf{I}-\mathbf{X}|^b {}_2F_1^\beta(c-a,b;c;-\mathbf{X}(\mathbf{I}-\mathbf{X})^{-1}) && (4.19) \\ &= |\mathbf{I}-\mathbf{X}|^{c-a-b} {}_2F_1^\beta(c-a,c-b;c;\mathbf{X}) && (4.20) \end{aligned}$$

for $||\mathbf{X}|| < 1$.

Remark 4.2. Observe that, for ${}_1F_0^\beta(a;\mathbf{X})$ the condition $\operatorname{Re}(a) > (m-1)\beta/2$ over a is determined by its integral representation (4.3). However ${}_1F_0^\beta(a;\mathbf{X})$ is easily seen to be analytic for all a and $||\mathbf{X}|| < 1$, see [27, p. 486]. Similarly, the conditions $\operatorname{Re}(c) > \operatorname{Re}(a) + (m-1)\beta/2 > (m-1)\beta$ over a and c given in Theorem 4.2, valid for Theorem 4.3 (4.18) too, are determined by the existence of the integral (4.17) and Remark 4.1. However, these conditions can be extended to other possible values if we use the inverse Laplace transformation to define ${}_1F_1^\beta(a,c;\mathbf{X})$. In this case ${}_1F_1^\beta(a,c;\mathbf{X})$ is valid for the arbitrary complex a, $\operatorname{Re}(c) > (m-1)\beta/2$ and $\mathbf{X} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, see [27, p. 487]. Also, the conditions $\operatorname{Re}(c) > \operatorname{Re}(a) + (m-1)\beta/2 > (m-1)\beta$ over a and c for (4.5) and Theorem 4.3(4.19) and (4.20) are determined by the absolutely convergence of the integral (4.5) and Remark 4.1. Again, these conditions about a and c can be extended to other possible values using the inverse Laplace transformation and the results for ${}_1F_1^\beta(a,c;\mathbf{X})$ obtained as described before, see [27, p. 489]. Finally, let us take into account that, for any analysis if the integral representation of ${}_1F_1^\beta(a,c;\mathbf{X})$ or ${}_2F_1^\beta(a,b;c;\mathbf{X})$ is not used explicitly, then the extended conditions for a and c could be considered.

Theorem 4.4. *Let* $\mathbf{X} \in \mathcal{L}_{m,n}^{\beta}$ *and* $\mathbf{H} = (\mathbf{H}_1|\mathbf{H}_2) \in \mathfrak{U}^{\beta}(n)$, $\mathbf{H}_1 \in \mathcal{V}_{m,n}^{\beta}$. *Then*

$$
\begin{aligned}
{}_0F_1^{\beta}(\beta n/2; \beta^2\mathbf{X}\mathbf{X}^*/4) &= \int_{\mathbf{H}\in\mathfrak{U}^{\beta}(n)} \operatorname{etr}(\beta\mathbf{X}^*\mathbf{H}_1)(d\mathbf{H}) &\quad (4.21)\\
&= \int_{\mathbf{H}_1\in\mathcal{V}_{m,n}^{\beta}} \operatorname{etr}(\beta\mathbf{X}^*\mathbf{H}_1)(d\mathbf{H}_1) &\quad (4.22)
\end{aligned}
$$

Proof. The proof is analogous to that given in the real case for [37, Theorem 7.4.1] and in the quaternion case by [34]. Alternative proofs can be established in an analogous form to those given by [28] and [27, p. 494-495]. For (4.22) it might be necessary to consider Lemma 9.5.3, p. 397 in [37]. □

On the basis of Theorem 3.1, we now discuss diverse integral properties of generalised hypergeometric functions, which contain as particular cases many of the results established above.

Theorem 4.5. *Assume* $p \le q$ *and* $\operatorname{Re}(a) > (m-1)\beta/2$ *and* $\mathbf{U} \in \mathfrak{S}_m^{\beta}$. *Then for* $\vartheta = \int_{z\in\mathfrak{P}_1^{\beta}} f(z)z^{am+k-1}dz < \infty$,

$$
\begin{aligned}
&\int_{\mathbf{X}\in\mathfrak{P}_m^{\beta}} f(\operatorname{tr}\mathbf{X}\mathbf{Z})\, {}_pF_q^{\beta}(a_1\cdots a_p; b_1\cdots b_q; \mathbf{X}\mathbf{U})|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X})\\
&\qquad = |\mathbf{Z}|^{-a}\Gamma_m^{\beta}[a] \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[a_1]_\kappa^{\beta}\cdots[a_p]_\kappa^{\beta}[a]_\kappa^{\beta}}{[b_1]_\kappa^{\beta}\cdots[b_q]_\kappa^{\beta}} \frac{C_\kappa^{\beta}(\mathbf{U}\mathbf{Z}^{-1})}{\Gamma[am+k]k!}\cdot\vartheta. \quad (4.23)
\end{aligned}
$$

When $p < q$, *the integral in (4.23) converges absolutely for all* $\mathbf{Z} \in \mathbf{\Phi}$. *When* $p = q$, *the integral converges absolutely for all* $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, *such that* $||(\operatorname{Re}(\mathbf{Z}))^{-1}|| < 1$.
Similarly, let $p \le q$, $\operatorname{Re}(a) > (m-1)\beta/2$ *and* $\mathbf{U} \in \mathfrak{S}_m^{\beta}$. *Then*

$$
\begin{aligned}
&\int_{\mathbf{X}\in\mathfrak{P}_m^{\beta}} f(\operatorname{tr}\mathbf{X}\mathbf{Z})\, {}_pF_q^{(m),\beta}(a_1\cdots a_p; b_1\cdots b_q; \mathbf{X}\mathbf{U}, \mathbf{Y})|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X})\\
&\qquad = |\mathbf{Z}|^{-a}\Gamma_m^{\beta}[a] \sum_{k=0}^{\infty}\sum_{\kappa} \frac{[a_1]_\kappa^{\beta}\cdots[a_p]_\kappa^{\beta}[a]_\kappa^{\beta}}{[b_1]_\kappa^{\beta}\cdots[b_q]_\kappa^{\beta}} \frac{C_\kappa^{\beta}(\mathbf{U}\mathbf{Z}^{-1})C_\kappa^{\beta}(\mathbf{Y})}{\Gamma[am+k]k!\ C_\kappa^{\beta}(\mathbf{I})}\cdot\vartheta. \quad (4.24)
\end{aligned}
$$

When $p < q$, *the integral in (4.24) converges absolutely for all* $\mathbf{Z} \in \mathbf{\Phi}$ *and* $\mathbf{Y} \in \mathfrak{S}_m^{\beta}$. *When* $p = q$, *the integral converges absolutely for all* $\mathbf{Z}$ *and* $\mathbf{Y} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, *such that* $||(\operatorname{Re}(\mathbf{Z}))^{-1}|| \cdot ||\mathbf{Y}|| < 1$.

Observe that if $f(\operatorname{tr}\mathbf{XZ}) = \operatorname{etr}\{-\mathbf{XZ}\}$ in Theorem 4.5 then we obtain (4.7) and (4.13).

Now, we propose the incomplete gamma and beta functions for normed division algebras.

Theorem 4.6. *Let* $\mathbf{\Lambda} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ *and* $\mathbf{\Omega} \in \mathbf{\Phi}$*. Then*

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{\Omega}} \operatorname{etr}\{-\mathbf{\Lambda X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X})$$
$$= \mathcal{B}_m^\beta[a,(m-1)\beta/2+1]|\mathbf{\Omega}|^a {}_1F_1^\beta(a; a+(m-1)\beta/2+1; -\mathbf{\Omega\Lambda}), \quad (4.25)$$

for $\operatorname{Re}(a) > (m-1)\beta/2$*. And, let* $\mathbf{0} < \mathbf{\Xi} < \mathbf{I}$*, then*

$$\int_{\mathbf{0}<\mathbf{Y}<\mathbf{\Xi}} |\mathbf{Y}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{Y}|^{b-(m-1)\beta/2-1}(d\mathbf{Y}) = \mathcal{B}_m^\beta[a,(m-1)\beta/2+1]$$
$$\times |\mathbf{\Xi}|^a {}_2F_1^\beta(a, -b+(m-1)\beta/2+1; a+(m-1)\beta/2+1; \mathbf{\Xi}), \quad (4.26)$$

for $\operatorname{Re}(a) > (m-1)\beta/2$ *and* $\operatorname{Re}(b) > (m-1)\beta/2$*.*

Proof. For (4.25), let us make the transformation $\mathbf{X} = \mathbf{\Omega}^{1/2}\mathbf{R}\mathbf{\Omega}^{1/2}$ and by applying Proposition 2.1 we have, $(d\mathbf{X}) = |\mathbf{\Omega}|^{(m-1)\beta/2+1}(d\mathbf{R})$, with $\mathbf{0} < \mathbf{R} < \mathbf{I}$. Then, expanding $\operatorname{etr}\{-\mathbf{\Lambda X}\}$ as a series of zonal spherical functions and integrating term by term using Corollary 3.6, the desired result is obtained. Similarly, (4.26) is proved by making the transformation $\mathbf{Y} = \mathbf{\Xi}^{1/2}\mathbf{R}\mathbf{\Xi}^{1/2}$ from where, applying the Proposition 2.1 we obtain that $(d\mathbf{X}) = |\mathbf{\Xi}|^{(m-1)\beta/2+1}(d\mathbf{R})$, with $\mathbf{0} < \mathbf{R} < \mathbf{I}$, expanding $|\mathbf{I}-\mathbf{X\Xi}|^{b-(m-1)\beta/2-1} = {}_1F_0^\beta(-b+(m-1)\beta/2+1; \mathbf{X\Xi})$ and integrating term by term using Corollary 3.6. □

Theorem 4.7. *Let* $\mathbf{\Lambda} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ *and* $\mathbf{\Omega} \in \mathbf{\Phi}$*. If* $r = a-(m-1)\beta/2-1$ *is a positive integer, then*

$$\int_{\mathbf{X}>\mathbf{\Omega}} \operatorname{etr}\{-\mathbf{\Lambda X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1}(d\mathbf{X})$$
$$= \Gamma_m^\beta[a]|\mathbf{\Lambda}|^{-a} \operatorname{etr}\{-\mathbf{\Lambda\Omega}\} \sum_{k=0}^{mr} \sum_{\kappa}{}^{*} \frac{C_\kappa^\beta(\mathbf{\Omega\Lambda})}{k!}, \quad (4.27)$$

where $\sum_\kappa^*$ *denotes summation over those partitions* $\kappa = (k_1, \ldots, k_m)$ *of* k *with* $k_1 \le r$*.*

Proof. Consider the transformation $\mathbf{X} = \mathbf{\Omega}^{1/2}(\mathbf{I}+\mathbf{R})\mathbf{\Omega}^{1/2}$ and applying Proposition 2.1 we have, $(d\mathbf{X}) = |\mathbf{\Omega}|^{(m-1)\beta/2+1}(d\mathbf{R})$, with $\mathbf{R} > \mathbf{0}$. Noting that $|\mathbf{I}+\mathbf{R}| = |\mathbf{R}||\mathbf{I}+\mathbf{R}^{-1}|$ and expanding $|\mathbf{I}+\mathbf{R}^{-1}|^{a-(m-1)\beta/2-1}$ in terms of zonal spherical functions, assuming that $r = a-(m-1)\beta/2-1$ is a positive integer we obtain

$$\begin{aligned}
&|\mathbf{I}+\mathbf{R}^{-1}|^{a-(m-1)\beta/2-1} \\
&\qquad = {}_1F_0^{\beta}(-a+(m-1)\beta/2+1; -\mathbf{R}^{-1}) \\
&\qquad = \sum_{k=0}^{mr}\sum_{\kappa}{}^{*}\frac{[-a+(m-1)\beta/2+1]_{\kappa}^{\beta}(-1)^k C_{\kappa}^{\beta}(\mathbf{R}^{-1})}{k!}
\end{aligned}$$

because $[-a+(m-1)\beta/2+1]_{\kappa}^{\beta} \equiv 0$ is any part of κ that is greater than r. The desired result is obtained by integrating term by term using Corollary 3.3. □

We end this section with some general results, which are useful in a variety of situations, which enable us to transform the density function of a matrix $\mathbf{X} \in \mathfrak{P}_m^{\beta}$ to the density function of its eigenvalues.

Theorem 4.8. *Let $\mathbf{X} \in \mathfrak{P}_m^{\beta}$ be a random matrix with density function $f(\mathbf{X})$. Then the joint density function of the eigenvalues $\lambda_1, \dots, \lambda_m$ of $\mathbf{X}$ is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^{\beta}[m\beta/2]}\prod_{i<j}^{m}(\lambda_i-\lambda_j)^{\beta}\int_{\mathbf{H}\in\mathfrak{U}^{\beta}(m)} f(\mathbf{HLH}^*)(d\mathbf{H}) \tag{4.28}$$

where $\mathbf{L} = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_1 > \cdots > \lambda_m > 0$, ϱ is defined in Proposition 2.3 and $(d\mathbf{H})$ is the normalised Haar measure.

Proof. The proof follows immediately from Proposition 2.3. □

5 Invariant Polynomials

In this section, we extend many of the properties of a class of homogeneous polynomials for normed division algebras of degrees k and t in the elements of matrices $\mathbf{X}$ and $\mathbf{Y} \in \mathfrak{S}_m^{\beta}$, respectively, see [7], [8], [4] and [5]; these are denoted as $C_{\phi}^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{Y})$. These homogeneous polynomials are invariant under the simultaneous transformations

$$\mathbf{X} \to \mathbf{H}^*\mathbf{XH}, \qquad \mathbf{Y} \to \mathbf{H}^*\mathbf{YH}, \qquad \mathbf{H} \in \mathfrak{U}^{\beta}(m).$$

The most important relationship of these polynomials is

$$\int_{\mathbf{H}\in\mathfrak{U}^{\beta}(m)} C_{\kappa}^{\beta}(\mathbf{AH}^{*}\mathbf{XH})C_{\tau}^{\beta}(\mathbf{BH}^{*}\mathbf{YH})(d\mathbf{H}) = \sum_{\phi\in\kappa.\tau}\frac{C_{\phi}^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B})C_{\phi}^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{Y})}{C_{\phi}^{\beta}(\mathbf{I})}, \tag{5.1}$$

where $(d\mathbf{H})$ is the normalised Haar measure and C_{κ}^{β}, C_{τ}^{β} and C_{ϕ}^{β} are zonal spherical functions indexed by ordered partitions κ, τ and ϕ of nonnegative integers k, t and $f=k+t$, respectively, into not more than m parts. $\phi\in\kappa.\tau$ denotes the irreducible representation of $GL(m,\mathfrak{F})$ indexed by 2ϕ that occurs in the decomposition of the Kronecker product $2\kappa\otimes 2\tau$ of the irreducible representations indexed by 2κ and 2τ, see [7] and [8].

In a similar way to the case of zonal spherical functions, let $\mathbf{A}=\mathbf{A}^{*}\mathbf{A}$ and $\mathbf{B}=\mathbf{B}^{*}\mathbf{B}$. For convenience of notation rather than strict adherence to rigor, we write $C_{\kappa}^{[\beta],\kappa,\tau}(\mathbf{XA},\mathbf{YB})$ or $C_{\kappa}^{[\beta],\kappa,\tau}(\mathbf{AX},\mathbf{BY})$ rather than $C_{\kappa}^{[\beta],\kappa,\tau}(\mathbf{AXA}^{*},\mathbf{BYB}^{*})$, even though $\mathbf{XA}$, $\mathbf{YB}$, $\mathbf{AX}$, or $\mathbf{BY}$ need not lie in $\mathfrak{S}_{m}^{\beta}$.

Some of the elementary properties and results on invariant polynomials are extended below:

Elementary properties of $C_{\phi}^{[\beta]\kappa,\tau}$.

Let $\mathbf{X}$ and $\mathbf{Y}\in\mathfrak{S}_{m}^{\beta}$, then

$$C_{\phi}^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{X})=\theta_{\phi}^{[\beta]\kappa,\tau}C_{\phi}^{\beta}(\mathbf{X}),\quad \text{where}\quad \theta_{\phi}^{[\beta]\kappa,\tau}=\frac{C_{\phi}^{[\beta]\kappa,\tau}(\mathbf{I},\mathbf{I})}{C_{\phi}^{\beta}(\mathbf{I})}. \tag{5.2}$$

$$C_{\phi}^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{Y})=\begin{cases}\dfrac{\theta_{\phi}^{[\beta]\kappa,\tau}C_{\phi}^{\beta}(\mathbf{I})}{C_{\kappa}^{\beta}(\mathbf{I})}C_{\kappa}^{\beta}(\mathbf{X}), & \text{for } \mathbf{Y}=\mathbf{I};\\ \dfrac{\theta_{\phi}^{[\beta]\kappa,\tau}C_{\phi}^{\beta}(\mathbf{I})}{C_{\tau}^{\beta}(\mathbf{I})}C_{\tau}^{\beta}(\mathbf{Y}), & \text{for } \mathbf{X}=\mathbf{I}.\end{cases} \tag{5.3}$$

$$C_{\phi}^{[\beta]\kappa,0}(\mathbf{X},\mathbf{Y})=C_{\kappa}^{\beta}(\mathbf{X}),\quad \text{and}\quad C_{\phi}^{[\beta]0,\tau}(\mathbf{X},\mathbf{Y})=C_{\tau}^{\beta}(\mathbf{Y}). \tag{5.4}$$

$$C_{\kappa}^{\beta}(\mathbf{X})C_{\tau}^{\beta}(\mathbf{Y})=\sum_{\phi\in\kappa.\tau}\theta_{\phi}^{[\beta]\kappa,\tau}C_{\phi}^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{Y}), \tag{5.5}$$

therefore,

$$(\operatorname{tr}\mathbf{X})^k(\operatorname{tr}\mathbf{Y})^t = \sum_{\kappa,\tau;\phi\in\kappa.\tau} \theta_\phi^{[\beta]\kappa,\tau} C_\phi^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{Y}). \tag{5.6}$$

From (5.2) and (5.5)

$$C_\kappa^\beta(\mathbf{X})C_\tau^\beta(\mathbf{X}) = \sum_{\phi\in\kappa.\tau} \left(\theta_\phi^{[\beta]\kappa,\tau}\right)^2 C_\phi^\beta(\mathbf{X}). \tag{5.7}$$

For constant a and b

$$C_\phi^{[\beta]\kappa,\tau}(a\mathbf{X}, b\mathbf{X}) = a^k b^t\ C_\phi^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{Y}). \tag{5.8}$$

The next expansion can be used to derive several useful results of invariant polynomials. From (5.1), (5.5) and (4.2) we obtain

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \operatorname{etr}\{\mathbf{A}\mathbf{H}^*\mathbf{X}\mathbf{H} + \mathbf{B}\mathbf{H}^*\mathbf{Y}\mathbf{H}\}(d\mathbf{H}) = \sum_{\kappa,\tau;\phi}^{\infty} \frac{C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B})C_\phi^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{Y})}{k!t!C_\phi^\beta(\mathbf{I})}, \tag{5.9}$$

where

$$\sum_{\kappa,\tau;\phi}^{\infty} = \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\sum_{\phi\in\kappa.\tau}.$$

From (5.9) we obtain,

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}^*\mathbf{H}^*\mathbf{X}\mathbf{H}\mathbf{A},\mathbf{B})(d\mathbf{H}) = \frac{C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}^*\mathbf{A},\mathbf{B})C_\kappa^\beta(\mathbf{X})}{C_\kappa^\beta(\mathbf{I})}, \tag{5.10}$$

analogously

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}^*\mathbf{H}^*\mathbf{Y}\mathbf{H}\mathbf{B})(d\mathbf{H}) = \frac{C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}^*\mathbf{B})C_\tau^\beta(\mathbf{Y})}{C_\tau^\beta(\mathbf{I})}, \tag{5.11}$$

Laplace transform.

For all $\mathbf{A}$ and $\mathbf{B}\in\mathfrak{S}_m^\beta$, $\mathbf{Z}\in\mathbf{\Phi}$

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{X}\mathbf{Z}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}\mathbf{X},\mathbf{B}\mathbf{X})(d\mathbf{X}) = \Gamma_m^\beta[a,\phi]|\mathbf{Z}|^{-a} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}\mathbf{Z}^{-1},\mathbf{B}\mathbf{Z}^{-1}). \tag{5.12}$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2 - (k+t)_m$. In particular

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AXA}^*, \mathbf{B})(d\mathbf{X}) = \Gamma_m^\beta[a,\kappa]|\mathbf{Z}|^{-a} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AZ}^{-1}\mathbf{A}^*, \mathbf{B}), \quad (5.13)$$

where $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$. And

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}, \mathbf{BXB}^*)(d\mathbf{X}) = \Gamma_m^\beta[a,\tau]|\mathbf{Z}|^{-a} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}, \mathbf{BZ}^{-1}\mathbf{B}^*), \quad (5.14)$$

with $\mathrm{Re}(a) > (m-1)\beta/2 - t_m$.

Similarly, for all $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^\beta$, $\mathbf{Z} \in \mathbf{\Phi}$,

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX}^{-1}, \mathbf{BX}^{-1})(d\mathbf{X}) = \Gamma_m^\beta[a,-\phi]|\mathbf{Z}|^{-a} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AZ}, \mathbf{BZ}), \quad (5.15)$$

where $\mathrm{Re}(a) > (m-1)\beta/2 + (k+t)_1$. In particular

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX}^{-1}\mathbf{A}^*, \mathbf{B})(d\mathbf{X}) = \Gamma_m^\beta[a,-\kappa]|\mathbf{Z}|^{-a} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AZA}^*, \mathbf{B}), \quad (5.16)$$

with $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$. And

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \mathrm{etr}\{-\mathbf{XZ}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}, \mathbf{BX}^{-1}\mathbf{B}^*)(d\mathbf{X}) = \Gamma_m^\beta[a,-\tau]|\mathbf{Z}|^{-a} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}, \mathbf{BZB}^*), \quad (5.17)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2 + t_1$.

Inverse Laplace transform.

Assume that $\mathbf{Z}$ and $\mathbf{X} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$, $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^\beta$ and $\mathrm{Re}(b) > b_0$. Then

$$\frac{\Gamma_m^\beta[b,\phi]2^{m(m-1)\beta/2}}{(2\pi i)^{m(m-1)\beta/2+m}} \int_{\mathbf{Z}-\mathbf{Z}_0\in\mathbf{\Phi}} \mathrm{etr}\{\mathbf{XZ}\}|\mathbf{Z}|^{-b} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AZ}^{-1}, \mathbf{BZ}^{-1})(d\mathbf{Z}) = |\mathbf{X}|^{b-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX}, \mathbf{BX}), \quad (5.18)$$

and

$$\frac{\Gamma_m^\beta[b,-\phi]2^{m(m-1)\beta/2}}{(2\pi i)^{m(m-1)\beta/2+m}}\int_{\mathbf{Z}-\mathbf{Z}_0\in\mathbf{\Phi}}\operatorname{etr}\{\mathbf{XZ}\}|\mathbf{Z}|^{-b}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AZ},\mathbf{BZ})(d\mathbf{Z})$$
$$= |\mathbf{X}|^{b-(m-1)\beta/2-1}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX}^{-1},\mathbf{BX}^{-1}). \quad (5.19)$$

Similar expressions are obtained for $C_\phi^{[\beta]\kappa,\tau}(\mathbf{AZ}^{-1},\mathbf{B})$ and $C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BZ}^{-1})$ from (5.18); and for $C_\phi^{[\beta]\kappa,\tau}(\mathbf{AZ},\mathbf{B})$ and $C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BZ})$ from (5.19).

Beta type I integrals.

For all $\mathbf{A}$ and $\mathbf{B}\in\mathfrak{S}_m^{\beta,\mathfrak{C}}$ and $\operatorname{Re}(b)>(m-1)\beta/2$,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}}|\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX},\mathbf{BX})(d\mathbf{X})$$
$$=\frac{\Gamma_m^\beta[a,\phi]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,\phi]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}), \quad (5.20)$$

valid for $\operatorname{Re}(a)>(m-1)\beta/2-(k+t)_m$. In particular

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}}|\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AXA}^*,\mathbf{B})(d\mathbf{X})$$
$$=\frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,\kappa]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AA}^*,\mathbf{B}), \quad (5.21)$$

with $\operatorname{Re}(a)>(m-1)\beta/2-k_m$. And

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}}|\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BXB}^*)(d\mathbf{X})$$
$$=\frac{\Gamma_m^\beta[a,\tau]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,\tau]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BB}^*). \quad (5.22)$$

where $\operatorname{Re}(a)>(m-1)\beta/2-t_m$. Another particular integral given in the real case by [7] is

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}}|\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\phi^{[\beta]\kappa,\tau}(\mathbf{X},\mathbf{I}-\mathbf{X})(d\mathbf{X})$$
$$=\frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}{\Gamma_m^\beta[a+b,\phi]}\theta_\phi^{[\beta]\kappa,\tau}C_\phi^\beta(\mathbf{I}), \quad (5.23)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$ and $\mathrm{Re}(b) > (m-1)\beta/2 - t_m$.

Analogously, for all $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ and $\mathrm{Re}(b) > (m-1)\beta/2$,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}\mathbf{X}^{-1},\mathbf{B}\mathbf{X}^{-1})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,-\phi]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,-\phi]} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}), \quad (5.24)$$

where $\mathrm{Re}(a) > (m-1)\beta/2 + (k+t)_1$. In particular

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}\mathbf{X}^{-1}\mathbf{A}^*,\mathbf{B})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,-\kappa]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,-\kappa]} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}\mathbf{A}^*,\mathbf{B}), \quad (5.25)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2 + k_1$. And

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}\mathbf{X}^{-1}\mathbf{B}^*)(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,-\tau]\Gamma_m^\beta[b]}{\Gamma_m^\beta[a+b,-\tau]} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}\mathbf{B}^*), \quad (5.26)$$

with $\mathrm{Re}(a) > (m-1)\beta/2 + t_1$.

Now, taking $b = (m-1)\beta/2 + 1 > (m-1)\beta/2$ in (5.20) and (5.24) we have the following results.

For all $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}\mathbf{X},\mathbf{B}\mathbf{X})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,\phi]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,\phi]} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}), \quad (5.27)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2 - (k+t)_m$. And,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{I}} |\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A}\mathbf{X}^{-1},\mathbf{B}\mathbf{X}^{-1})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,-\phi]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,-\phi]} C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}), \quad (5.28)$$

where $\mathrm{Re}(a) > (m-1)\beta/2 + (k+t)_1$.

Beta type II integrals.

For all $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$,

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX},\mathbf{BX})(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,\phi]\Gamma_m^\beta[b,-\phi]}{\Gamma_m^\beta[a+b]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}), \tag{5.29}$$

with $\mathrm{Re}(a) > (m-1)\beta/2-(k+t)_m$ and $\mathrm{Re}(b) > (m-1)\beta/2+(k+t)_1$. In particular

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AXA}^*,\mathbf{B})(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,-\kappa]}{\Gamma_m^\beta[a+b]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AA}^*,\mathbf{B}), \tag{5.30}$$

such that $\mathrm{Re}(a) > (m-1)\beta/2-k_m$ and $\mathrm{Re}(b) > (m-1)\beta/2+k_1$. And

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BXB}^*)(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,\tau]\Gamma_m^\beta[b,-\tau]}{\Gamma_m^\beta[a+b]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BB}^*), \tag{5.31}$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2-t_m$ and $\mathrm{Re}(b) > (m-1)\beta/2+t_1$.

In a similar way, for all $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$,

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX}^{-1},\mathbf{BX}^{-1})(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,-\phi]\Gamma_m^\beta[b,\phi]}{\Gamma_m^\beta[a+b]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{B}) \tag{5.32}$$

where $\mathrm{Re}(a) > (m-1)\beta/2+(k+t)_1$ and $\mathrm{Re}(b) > (m-1)\beta/2-(k+t)_m$. In particular

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX}^{-1}\mathbf{A}^*,\mathbf{B})(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,-\kappa]\Gamma_m^\beta[b,\kappa]}{\Gamma_m^\beta[a+b]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{AA}^*,\mathbf{B}), \tag{5.33}$$

with $\mathrm{Re}(a) > (m-1)\beta/2+k_1$ and $\mathrm{Re}(b) > (m-1)\beta/2-k_m$. And

$$\int_{\mathbf{X}\in\mathfrak{P}_m^\beta} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}+\mathbf{X}|^{-(a+b)}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BX}^{-1}\mathbf{B}^*)(d\mathbf{X})$$
$$= \frac{\Gamma_m^\beta[a,-\tau]\Gamma_m^\beta[b,\tau]}{\Gamma_m^\beta[a+b]}C_\phi^{[\beta]\kappa,\tau}(\mathbf{A},\mathbf{BB}^*), \tag{5.34}$$

such that $\mathrm{Re}(a) > (m-1)\beta/2 + t_1$ and $\mathrm{Re}(b) > (m-1)\beta/2 - t_m$.

Incomplete gamma and beta functions.

First consider the following results

For all $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^\beta$ and $\mathbf{0} < \mathbf{\Xi} < \mathbf{I}$,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{\Xi}} |\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AX}, \mathbf{BX})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,\phi]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,\phi]} |\mathbf{\Omega}|^a C_\phi^{[\beta]\kappa,\tau}(\mathbf{A\Xi}, \mathbf{B\Xi}). \quad (5.35)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2 - (k+t)_m$. And

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{\Xi}} |\mathbf{X}|^{a-(m-1)\beta/2-1} C_\phi^{[\beta]\kappa,\tau}(\mathbf{AXA}^*, \mathbf{B})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1,\kappa]} |\mathbf{\Omega}|^a C_\phi^{[\beta]\kappa,\tau}(\mathbf{A\Xi A}^*, \mathbf{B}). \quad (5.36)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2 - k_m$.

The next result is obtained immediately, expanding $\mathrm{etr}\{-\mathbf{XA}\}$ in terms of zonal spherical functions, making use of the property (5.5) and integrating term by term using (5.35). Thus, for all $\mathbf{A}$ and $\mathbf{B} \in \mathfrak{S}_m^\beta$ and $\mathbf{\Omega} \in \mathbf{\Phi}$,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{\Omega}} \mathrm{etr}\{-\mathbf{XA}\} |\mathbf{X}|^{a-(m-1)\beta/2-1} C_\tau^\beta(\mathbf{BX})(d\mathbf{X}) = \frac{\Gamma_m^\beta[a]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1]} |\mathbf{\Omega}|^a \times \sum_{k=0}^\infty \sum_{\kappa;\phi\in\kappa.\tau} \frac{[a]_\phi^\beta \theta_\phi^{[\beta]\kappa,\tau} C_\phi^{[\beta]\kappa,\tau}(-\mathbf{A\Omega}, \mathbf{B\Omega})}{k![a+(m-1)\beta/2+1]_\phi^\beta}. \quad (5.37)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2$.

Similarly, we expand $|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}$ in terms of Jack polynomials, make use of the property (5.5) and integrate term by term using

(5.35). For all $\mathbf{A} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ and $\mathbf{0} < \mathbf{\Xi} < \mathbf{I}$,

$$\int_{\mathbf{0}<\mathbf{X}<\mathbf{\Xi}} |\mathbf{X}|^{a-(m-1)\beta/2-1}|\mathbf{I}-\mathbf{X}|^{b-(m-1)\beta/2-1}C_\tau^\beta(\mathbf{AX})(d\mathbf{X})$$

$$= \frac{\Gamma_m^\beta[a]\Gamma_m^\beta[(m-1)\beta/2+1]}{\Gamma_m^\beta[a+(m-1)\beta/2+1]}$$

$$\times\ |\mathbf{\Xi}|^a \sum_{k=0}^{\infty}\sum_{\kappa;\phi\in\kappa.\tau} \frac{[-b+(m-1)\beta/2+1]_\kappa^\beta [a]_\phi^\beta \theta_\phi^{[\beta]\kappa,\tau} C_\phi^{[\beta]\kappa,\tau}(\mathbf{\Xi},\mathbf{A\Xi})}{k![a+(m-1)\beta/2+1]_\phi^\beta}. \quad (5.38)$$

valid for $\mathrm{Re}(a) > (m-1)\beta/2$.

6 Application

As an application, in this section we found the joint density eigenvalue of the central Wishart distribution for normed division algebras, and also derived the largest and smallest eigenvalue distributions. First, from [12] let us consider the following definitions.

Definition 6.1. Let $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$ be a random matrix. Then $\mathbf{X}$ is said to have a matrix variate normal distribution $\mathbf{X} \sim \mathcal{N}_{n\times m}^\beta(\boldsymbol{\mu},\mathbf{\Sigma},\mathbf{\Theta})$, of mean $\boldsymbol{\mu}$ and $\mathrm{Cov}(\mathrm{vec}\,\mathbf{X}^T) = \mathbf{\Theta}\otimes\mathbf{\Sigma}$, if its density function is given by

$$\frac{1}{(2\pi\beta^{-1})^{\beta mn/2}|\mathbf{\Sigma}|^{\beta n/2}|\mathbf{\Theta}|^{\beta m/2}}\,\mathrm{etr}\left\{-\frac{\beta}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{X}-\boldsymbol{\mu})\right\}.$$

Also

Definition 6.2. Let $\mathbf{X} \in \mathcal{L}_{m,n}^\beta$ with distribution $\mathbf{X} \sim \mathcal{N}_{n\times m}^\beta(\mathbf{0},\mathbf{\Sigma},\mathbf{I}_n)$ and define $\mathbf{S} = \mathbf{X}^*\mathbf{X}$, then $\mathbf{S}$ is said to have a central Wishart distribution $\mathbf{S} \sim \mathcal{W}_m^\beta(n,\mathbf{\Sigma})$ with n degrees of freedom and parameter $\mathbf{\Sigma}$. Moreover, its density function is given by

$$\frac{1}{(2\beta^{-1})^{\beta mn/2}\,\Gamma_m^\beta[\beta n/2]|\mathbf{\Sigma}|^{\beta n/2}}|\mathbf{S}|^{\beta(n-m+1)/2-1}\,\mathrm{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{S}/2\}$$

with $n \geq (m-1)\beta$.

Therefore, from (4.28) and (4.12) the joint density of the eigenvalues, $\lambda_1 > \cdots > \lambda_m > 0$, of $\mathbf{S}$ is

$$\frac{\pi^{m^2\beta/2+\varrho}}{(2\beta^{-1})^{\beta mn/2}\,\Gamma_m^\beta[\beta n/2]\Gamma_m^\beta[\beta m/2]|\mathbf{\Sigma}|^{\beta n/2}}$$

$$\times\prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1} \times \prod_{i<j}^m(\lambda_i-\lambda_j)^\beta\,{}_0F_0^\beta(-\beta\mathbf{\Sigma}^{-1}/2,\mathbf{L})$$

where $\mathbf{L} = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$.

In addition, as an immediate consequence of Theorems 4.6 and 4.7 we obtain the following result.

Theorem 6.1. *Let $\mathbf{S} \sim \mathcal{W}_m^\beta(n, \mathbf{\Sigma})$ and $\mathbf{\Omega} \in \mathbf{\Phi}$, then*

$$P(\mathbf{S} < \mathbf{\Omega}) = \frac{\Gamma_m^\beta[(m-1)\beta/2+1]}{(2\beta^{-1})^{\beta mn/2}\,\Gamma_m^\beta[(n+m-1)\beta/2+1]} \frac{|\mathbf{\Omega}|^{\beta n/2}}{|\mathbf{\Sigma}|^{\beta n/2}} \times {}_1F_1^\beta(\beta n/2; (n+m-1)\beta/2+1; -\beta\mathbf{\Omega}\mathbf{\Sigma}^{-1}/2), \tag{6.1}$$

valid for $\mathrm{Re}(n) > (m-1)\beta$. *And if $r = (n-m+1)\beta/2 - 1$ is a positive integer, then*

$$P(\mathbf{S} < \mathbf{\Omega}) = \mathrm{etr}\{-\beta\mathbf{\Omega}\mathbf{\Sigma}^{-1}/2\} \sum_{k=0}^{mr} \sum_{\kappa}{}^{*} \frac{C_\kappa^\beta(\beta\mathbf{\Omega}\mathbf{\Sigma}^{-1}/2)}{k!}, \tag{6.2}$$

where $\sum_\kappa^*$ denotes summation over those partitions $\kappa = (k_1, \dots, k_m)$ of k with $k_1 \leq r$.

Observing that if $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathbf{S}$, respectively, then the inequalities $\lambda_{\max} < x$ and $\lambda_{\min} > y$ are equivalent to $\mathbf{S} < x\mathbf{I}$ and $\mathbf{S} > y\mathbf{I}$, respectively and the following result is obtained.

Corollary 6.1. *Assume that $\mathbf{S} \sim \mathcal{W}_m^\beta(n, \mathbf{\Sigma})$ and $x > 0$. Then*

$$P(\lambda_{\max} < x) = \frac{\Gamma_m^\beta[(m-1)\beta/2+1]}{(2\beta^{-1})^{\beta mn/2}\,\Gamma_m^\beta[(n+m-1)\beta/2+1]} \frac{x^{\beta mn/2}}{|\mathbf{\Sigma}|^{\beta n/2}} \times {}_1F_1^\beta(\beta n/2; (n+m-1)\beta/2+1; -\beta x\mathbf{\Sigma}^{-1}/2), \tag{6.3}$$

valid for $\mathrm{Re}(n) > (m-1)\beta$. *And if $r = (n-m+1)\beta/2 - 1$ is a positive integer and $y > 0$, then*

$$P(\lambda_{\min} < y) = 1 - \mathrm{etr}\{-\beta y\mathbf{\Sigma}^{-1}/2\} \sum_{k=0}^{mr} \sum_{\kappa}{}^{*} \frac{C_\kappa^\beta(\beta y\mathbf{\Sigma}^{-1}/2)}{k!}, \tag{6.4}$$

where $\sum_\kappa^*$ denotes summation over those partitions $\kappa = (k_1, \dots, k_m)$ of k with $k_1 \leq r$.

As a numerical example we plot the distribution function of λ_{max} on Figure 1 and the distribution function of λ_{min} on Figure 2. First note that applying the generalised Kummer relation (4.18) in (6.3) we obtain

$$P(\lambda_{\max} < x) = \frac{\Gamma_m^\beta[(m-1)\beta/2+1]\,\mathrm{etr}\{-\beta x\mathbf{\Sigma}^{-1}/2\}}{(2\beta^{-1})^{\beta mn/2}\,\Gamma_m^\beta[(n+m-1)\beta/2+1]} \frac{x^{\beta mn/2}}{|\mathbf{\Sigma}|^{\beta n/2}} \times {}_1F_1^\beta((m-1)\beta/2+1; (n+m-1)\beta/2+1; \beta x\mathbf{\Sigma}^{-1}/2).$$

Currently, many other applications of the results in this work are being studied in the context of shape theory, random matrices and multivariate statistical analysis, both by the present authors and by others.

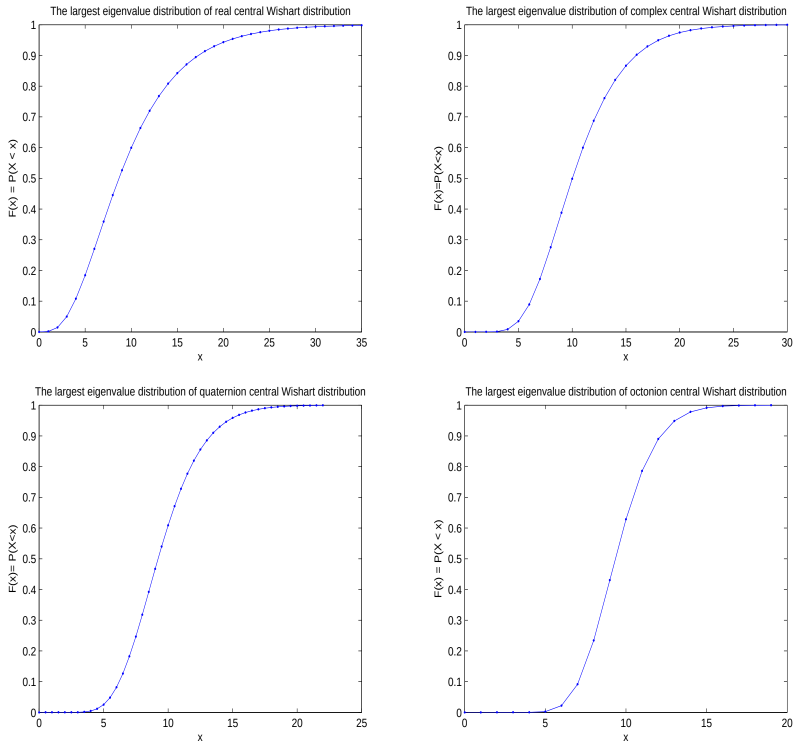

Figure 1: Distribution functions of $\lambda_{\max}$ of $\mathcal{W}_2^{\beta}(4, \mathrm{diag}(1,2))$, $\beta = 1, 2, 4$ and 8.

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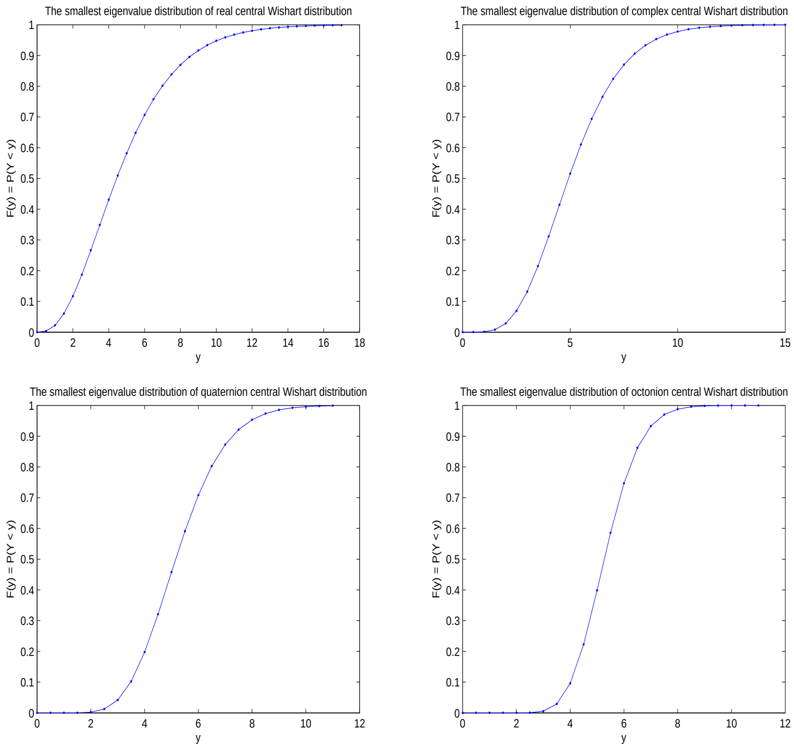

Figure 2: Distribution functions of $\lambda_{\min}$ of $\mathcal{W}_2^{\beta}(7, \mathrm{diag}(1,2))$, $\beta = 1, 2, 4$ and 8.

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