JIRSS (2013)
Vol. 12, No. 2, pp 235-252

Asymptotic Efficiencies of the MLE Based on Bivariate Record Values from Bivariate Normal Distribution

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Abstract. Maximum likelihood (ML) estimation based on bivariate record data is considered as the general inference problem. Assume that the process of observing k records is repeated m times, independently. The asymptotic properties including consistency and asymptotic normality of the Maximum Likelihood (ML) estimates of parameters of the underlying distribution is then established, when m is large enough. The bivariate normal distribution is considered as an highly applicable example in order to estimate the parameter $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2)$ by ML method of estimation based on mk bivariate record data. Asymptotic variances of the ML estimators are calculated by deriving the Fisher information matrix about θ contained in the vector of the first k bivariate record data. As another application, we concerned the problem of "breaking boards" of Glick (1978, *Amer. Math. Monthly*, 85, 2-26) by considering three different sampling schemes of breaking boards and we computed the relative asymptotic efficiencies of ML estimators based on these three types of data.

Keywords. Additivity, bivariate distribution, Fisher information matrix, inverse sampling, Jensen's inequality.

MSC: 62F12, 62N02, 62F10.

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Received: April 2012; Accepted: September 2012

1 Introduction

In a sequence of pair-wise random variables, the values of one variable that exceed or fall below the previous values of that variable are called record values and the corresponding value of other variable to each exceeding value is the concomitant of that record value. The bivariate sequence is then termed the sequence of bivariate record values. The reader is referred to Arnold et al. (1998) for more details on univariate and bivariate record data.

Estimation of parameters of bivariate normal distribution based on bivariate ordered random variables is concerned by a few authors. Harrell and Sen (1979), Qinying He and Nagaraja (2009) and the references therein, consider the problem of estimation based on bivariate order statistics under the bivariate normal distribution. Chacko and Thomas (2008) obtained best linear unbiased estimators of the parameters of the bivariate normal distribution in terms of only concomitants of record values.

The problem of deriving Fisher information contained in bivariate record values was studied by Amini and Ahmadi (2007, 2008). Amini and Ahmadi (2009) studied the additional information by considering inter-record times.

In this paper we consider ML estimation based on bivariate record data. Since for a single record-breaking sample, the number of record data are rare, it is inapplicable to investigating the asymptotic properties of the estimators when the number of records in a sequence gets large. In nonparametric setting, Samaniego and Whitaker (1988) extend the single sample results to the multi-sample case and studied the consistency of the nonparametric maximum-likelihood estimator of the sampling distribution.

In this sampling scheme, one repeat an inverse sampling scheme for achieving record data m times. The asymptotic properties, then, can be concerned by letting m to get large. A balanced scheme includes m independent sequences each with the first k record data. We show that in such a scheme, the ML estimators are consistent and asymptotically normal as m tends to infinity. One of the most important bivariate distributions in statistical inference is the bivariate normal distribution. As an application of the proposed estimation we estimate the parameter $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2)$ by ML estimation method based on mk bivariate record data. Asymptotic variances of the ML estimators are calculated by deriving the Fisher information matrix about θ contained in the vector of the first k bivariate record data.

As another application of the aforementioned inferential problem consider life testing problems in which testing of an item is destructive and costly. If the items are expensive, one can set up the experiment so that only the low life units are destroyed. As an example, one may consider the example of testing the breaking strength of wooden beams cited in Gulati and Padgett (1994 a, b) and Glick (1978), in which beams are replaced when they do not break until the pressure reaches the minimum previously observed breaking time. In other words, only the lower record values are observed. Now, suppose that we want to fit a model for the relationship between the breaking strength as the response variable and some other characteristics of wooden beams as predictors (such as elasticity). Here our only observations are the lower records of breaking strength and their associated concomitants. In such situations the problem of estimation based on record values and their concomitants is in a high degree of importance. Suppose we want to observe the "strength" (X) and "elasticity" (Y) of mk beams by a destructive life test. One can consider, the following three different sampling schemes for testing the wooden beams:

A: Test the beams by replacing them when they do not break until the pressure reaches the minimum previously observed breaking time. Continue the experiment until breaking k beams. Repeat the whole experiment m times.

B: Perform test **A** and also save the number of beams remaining unbroken between the i^{th} and the $(i+1)^{\text{th}}$ breaking beams in each replication.

C: Simply test mk beams until breaking.

Test **A** and **B** are similar tests, except that we observe an additional variable which are record times. Indeed, in test **B**, we consider the additional information that the strength of the beams remaining unbroken between the i^{th} and the $(i+1)^{\text{th}}$ breaking beams is greater than the strength of the i^{th} breaking beam. In test **C** the experimenter observe a bivariate random sample of size mk. Although, in tests **A** and **B**, more beams are tested, the number of destroyed beams in the three tests is equal. However, the tests **A** and **B** are more economical tests, since the more "low life" items are destroyed.

The question is that "which sampling scheme leads to more precise estimators of population parameters?". Here, the relative asymptotic efficiencies of the ML estimators, based on the above three types of data, can be calculated using the Fisher information matrix about θ contained in the vector of the first k bivariate record data.

The paper is organized as follows. Section 2 contains some notations. In Section 3, the main results of ML estimation and limit theorems are proposed. Section 4, concerns the ML estimation of the parameters of the bivariate normal distribution, using bivariate record values. Finally, we calculate the Fisher information matrices and asymptotic relative efficiencies.

2 Notations

Let $\{(X_i, Y_i), i \geq 1\}$ be a sequence of i.i.d. pair-wise random variables with the absolutely continuous cumulative distribution function (cdf) $F_{X,Y}(x, y; \theta)$ and the corresponding pdf $f_{X,Y}(x, y; \theta)$. Also $f_X(x; \theta)$ and $F_X(x; \theta)$ denote the marginal pdf and cdf of X, respectively and $\bar{F}_X(x; \theta) = 1 - F_X(x; \theta)$.

The sequence of lower records and their concomitants is defined as

$$(R_i, R_{[i]}) = (X_{T_i}, Y_{T_i}), \quad i \geq 1,$$

where $T_1 = 1$ with probability one and for $i \geq 2$, $T_i = \min\{j : j > T_{i-1},\ X_j < X_{T_{i-1}}\}$.

Suppose

$$\Delta_i = T_{i+1} - T_i - 1,\ i = 1, 2, \ldots, k-1,\ \ \Delta_k = 0,$$

are the number of trials needed to obtain new records, which are called inter-record times. Let us denote

$$\mathbf{R}(k) = (R_1, \ldots, R_k),\ \boldsymbol{\Delta}(k) = (\Delta_1, \ldots, \Delta_k), \mathbf{C}(k) = (R_{[1]}, \ldots, R_{[k]}),$$

$$\mathcal{R}_m = \{(R_{1(j)}, \ldots, R_{k(j)}),\ \ j = 1, \ldots, m\},$$

$$\mathcal{C}_m = \{(R_{[1](j)}, \ldots, R_{[k](j)}),\ \ j = 1, \ldots, m\},$$

and

$$\nabla_m = \{(\Delta_{1(j)}, \ldots, \Delta_{k(j)}),\ \ j = 1, \ldots, m\},$$

where $(R_i(j), R_{[i](j)}, \Delta_{i(j)})$ stands for the ith record data in the jth sequence, $i = 1, \ldots, k,\ \ j = 1, \ldots, m$.

The joint pdf of $\mathcal{R}_m$ and $\mathcal{C}_m$ is (see Arnold et al., 1998)

$$f_{\mathcal{R}_m, \mathcal{C}_m}(\mathbf{r}, \mathbf{c}; \theta) = \prod_{j=1}^{m} \prod_{i=1}^{k} f_{X,Y}(r_{i(j)}, c_{i(j)}; \theta) / \prod_{i=1}^{k-1} [F_X(r_{i(j)}; \theta)]. \qquad (1)$$

Using (1) the joint pdf of $\mathcal{R}_m$, $\mathcal{C}_m$ and ∇_m is given by

$$f_{\mathcal{R}_m,\mathcal{C}_m,\nabla_m}(\mathbf{r},\mathbf{c},\delta;\theta) = \prod_{j=1}^{m}\prod_{i=1}^{k} f_{X,Y}(r_{i(j)}, c_{i(j)};\theta)\{\bar{F}_X(r_{i(j)};\theta)\}^{\delta_{i(j)}}. \quad (2)$$

So, the conditional probability mass function of ∇_m given $(\mathcal{R}_m, \mathcal{C}_m)$ is given by

$$f_{\nabla_m|\mathcal{R}_m,\mathcal{C}_m}(\delta|\mathbf{r},\mathbf{c};\theta) = \prod_{j=1}^{m}\prod_{i=1}^{k-1}[\bar{F}_X(r_{i(j)};\theta)]^{\delta_{i(j)}} F_X(r_{i(j)};\theta). \quad (3)$$

We consider lower bivariate records to derive the results of this paper. Similar results also hold for the case of upper bivariate record data. Hereafter, we will call lower records, simply, records.

3 Main Results

Suppose that the assumptions of Section 2 hold. Also, let $\theta = (\theta_1, \ldots, \theta_l) \in \Theta \subset \mathcal{R}^l$ and the data involves $(\mathcal{R}_m, \mathcal{C}_m, \nabla_m)$. Using (1) and (2), the likelihood equations based on bivariate records only are

$$\sum_{j=1}^{m}\sum_{i=1}^{k} \frac{\partial}{\partial\theta_s} \log f_{X,Y}(r_{i(j)}, c_{i(j)};\theta) - \sum_{j=1}^{m}\sum_{i=1}^{k-1} \frac{\partial}{\partial\theta_s} \log F(r_{i(j)};\theta) = 0, \; s = 1, \ldots, l.$$

Also the likelihood equations based on bivariate records and inter-record times are given by

$$\sum_{j=1}^{m}\sum_{i=1}^{k} \frac{\partial}{\partial\theta_s} \log f_{X,Y}(r_{i(j)}, c_{i(j)};\theta) + \sum_{j=1}^{m}\sum_{i=1}^{k} \delta_{i(j)} \frac{\partial}{\partial\theta_s} \log \bar{F}(r_{i(j)};\theta) = 0, \; s = 1, \ldots, l.$$

The following theorems establish consistency and asymptotic normality of the roots of the above equations (if exist), as m tends to infinity.

Theorem 3.1. *Let $\hat{\theta}_m$ be the MLE of θ based on m independent sequences of the first k bivariate record values, then $\hat{\theta}_m$ is a consistent estimator for θ as $m \to \infty$.*

Proof. Let θ_0 be the value of θ. Since $\hat{\theta}_m$ is the ML estimator of θ, we have

$$\log\ f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\hat{\theta}_m) \geq \log\ f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta), \quad \text{for all } \theta \in \Theta. \quad (4)$$

On the other hand by Jensen's inequality we have

$$\mathrm{E}_{\theta_0}\left(\log \frac{f_{\mathbf{R},\mathbf{C}}(\mathbf{r},\mathbf{s};\theta)}{f_{\mathbf{R},\mathbf{C}}(\mathbf{r},\mathbf{s};\theta_0)}\right) \le \log \mathrm{E}_{\theta_0}\left(\frac{f_{\mathbf{R},\mathbf{C}}(\mathbf{r},\mathbf{s};\theta)}{f_{\mathbf{R},\mathbf{C}}(\mathbf{r},\mathbf{s};\theta_0)}\right) = 0.$$

So

$$\mathrm{E}_{\theta_0}\left(\log\ f_{\mathbf{R},\mathbf{C}}(\mathbf{r},\mathbf{s};\theta)\right) \le \mathrm{E}_{\theta_0}\left(\log\ f_{\mathbf{R},\mathbf{C}}(\mathbf{r},\mathbf{s};\theta_0)\right).$$

Since for all $\theta \in \Theta$ $\frac{1}{m}\log\ f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta)$ tends almost everywhere to $\mathrm{E}_{\theta_0}\left(\log\ f_{\mathbf{R},\mathbf{C}}(\mathbf{r},\mathbf{s};\theta)\right)$ as $m \to \infty$, hence with probability one and for large enough m, we have

$$\frac{1}{m}\log\ f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta) \le \frac{1}{m}\log\ f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta_0), \quad \text{for all } \theta \in \Theta. \tag{5}$$

Setting $\theta = \theta_0$ in (4) and $\theta = \hat{\theta}_m$ in (5), it is deduced that $\hat{\theta}_m$ tends to θ_0 with probability one, as $m \to \infty$. □

Theorem 3.2. *Let $\hat{\theta}_m$ be as in Theorem 3.1, then the asymptotic distribution of $\sqrt{m}(\hat{\theta}_m - \theta)$ is $\mathcal{N}_l(\mathbf{0}\ ,\ I^{-1}_{\mathbf{R}(k),\mathbf{C}(k)}(\theta))$ as $m \to \infty$, where $I^{-1}_{\mathbf{R}(k),\mathbf{C}(k)}(\theta)$ is the inverse of Fisher information matrix in $(\mathbf{R}(k), \mathbf{C}(k))$.*

Proof. The Taylor's expansion of $\frac{\partial}{\partial\theta_r}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta)\Big|_{\theta=\hat{\theta}}$ for $r = 1, \ldots, l$ around an arbitrarily θ_1 is equal to

$$\begin{aligned}
&\frac{\partial}{\partial\theta_r}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta)\Big|_{\theta=\hat{\theta}} \\
&= \frac{\partial}{\partial\theta_r}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta)\Big|_{\theta=\theta_1} \\
&\quad + \sum_{s=1}^{l}(\hat{\theta}_s - \theta_1)\ \frac{\partial^2}{\partial\theta_r\partial\theta_s}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta)\Big|_{\theta=\theta^*},
\end{aligned}$$

where θ^* is a sequence in Θ which tends in probability to θ_1. The left hand side of the above equality is equal to zero and hence

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta)\Big|_{\theta=\theta_1} \\
&= \sum_{s=1}^{l}(\hat{\theta}_s - \theta_1)\ \frac{-\partial^2}{\partial\theta\partial\theta_s}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m, \varsigma_m;\theta)\Big|_{\theta=\theta^*} \\
&= m(\hat{\theta} - \theta_1) I_{\mathbf{R}(k),\mathbf{C}(k)}(\theta_1).
\end{aligned}$$

On the other hand, $\frac{\partial}{\partial\theta}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m,\varsigma_m;\theta)\big|_{\theta=\theta_1}$ has a mean equal to zero and a variance equal to $mI_{\mathbf{R}(k),\mathbf{C}(k)}(\theta_1)$. So, by strong low of large numbers, we have

$$\begin{aligned}&\frac{1}{\sqrt{m}}\frac{\partial}{\partial\theta}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m,\varsigma_m;\theta)\Big|_{\theta=\theta_1}\\ &= \sqrt{m}\left(\frac{1}{m}\frac{\partial}{\partial\theta}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m,\varsigma_m;\theta)\Big|_{\theta=\theta_1}-0\right)\end{aligned}$$

has an asymptotic distribution $\mathcal{N}_l(\mathbf{0}\ ,\ I_{\mathbf{R}(k),\mathbf{C}(k)}(\theta_1))$ as $m\to\infty$. Hence,

$$\sqrt{m}(\hat{\theta}_m-\theta)=I^{-1}_{\mathbf{R}(k),\mathbf{C}(k)}(\theta)\frac{1}{\sqrt{m}}\frac{\partial}{\partial\theta}\log f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m,\varsigma_m;\theta)$$

has an asymptotic distribution $\mathcal{N}_l(\mathbf{0}\ ,\ I^{-1}_{\mathbf{R}(k),\mathbf{C}(k)}(\theta))$ as $m\to\infty$. $\square$

Remark 3.1. Similar asymptotic results as presented in Theorems 3.1 and 3.2, are hold for ML estimators based on bivariate records and their inter-record times by replacing $f_{\mathcal{R}_m,\mathcal{C}_m}(\varrho_m,\varsigma_m;\theta)$ with $f_{\mathcal{R}_m,\mathcal{C}_m,\nabla_m}(\varrho_m,\varsigma_m,\delta_m;\theta)$ and $I_{\mathbf{R},\mathbf{C}}(\theta)$ with $I_{\mathbf{R},\mathbf{C},\mathbf{\Delta}}(\theta)$ in Theorems 3.1 and 3.2.

4 Bivariate Normal Parameter Estimation

Suppose that m independent sequences each with k bivariate records are observed and the sampling distribution is bivariate normal distribution with parameter $\theta=(\mu_1,\sigma_1,\mu_2,\sigma_2,\rho)$. Let $(R_{i(j)},R_{[i](j)})$ denote the ith bivariate record in the jth sequence. Also, denote the standard hazard rate function and standard reverse hazard rate function of X, respectively, by $h_0(x)=f_X(x;\theta_0)/\bar{F}_X(x,\theta_0)$ and $r_0(x)=f_X(x;\theta_0)/F_X(x,\theta_0)$, in which $\theta_0=(0,1,,\mu_2,\sigma_2,\rho)$. Our aim is to obtain the MLE of θ based on bivariate records. The likelihood equations for bivariate record values and record times are as follows:

$$\begin{aligned}\sum_{j=1}^{m}\sum_{i=1}^{k}\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right) &- \frac{\rho}{\sigma_1}\sum_{j=1}^{m}\sum_{i=1}^{k}\left(\frac{R_{[i](j)}-\mu_2}{\sigma_2}\right)\\ &- \frac{1-\rho^2}{\sigma_1}\sum_{j=1}^{m}\sum_{i=1}^{k}\Delta_i h_0\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right)=0, \quad (6)\end{aligned}$$

$$\sum_{j=1}^{m}\sum_{i=1}^{k} R_{[i](j)} - mk\mu_2 - \frac{\rho\sigma_2}{\sigma_1}\left(\sum_{j=1}^{m}\sum_{i=1}^{k} R_{i(j)} - mk\mu_1\right) = 0,$$

$$\rho\left(mk - \frac{\sum_{j=1}^{m}\sum_{i=1}^{k}(R_{i(j)}-\mu_1)^2}{\sigma_1^2} - \frac{\sum_{j=1}^{m}\sum_{i=1}^{k}(R_{[i](j)}-\mu_2)^2}{\sigma_2^2}\right) + C^*(\theta)(\rho^2+1) - mk\rho^3 = 0,$$

where $C^*(\theta) = \sum_{j=1}^{m}\sum_{i=1}^{k}\left(R_{i(j)}-\mu_1\right)\left(R_{[i](j)}-\mu_2\right)/(\sigma_1\sigma_2)$,

$$mk(1-\rho^2) - \sum_{j=1}^{m}\sum_{i=1}^{k}\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right)^2 + \rho C^*(\theta) + (1-\rho^2)\sum_{j=1}^{m}\sum_{i=1}^{k}\frac{\Delta_i(R_{i(j)}-\mu_1)}{\sigma_1}h_0\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right) = 0, \tag{7}$$

and

$$mk(1-\rho^2) - \sum_{j=1}^{m}\sum_{i=1}^{k}\left(\frac{R_{[i](j)}-\mu_2}{\sigma_2}\right)^2 + \rho C^*(\theta) = 0.$$

For the case of bivariate record values without record times, the above equations remain true except that (6) and (7) are replaced, respectively, with

$$\sum_{j=1}^{m}\sum_{i=1}^{k}\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right) - \frac{\rho}{\sigma_1}\sum_{j=1}^{m}\sum_{i=1}^{k}\left(\frac{R_{[i](j)}-\mu_2}{\sigma_2}\right) - \frac{1-\rho^2}{\sigma_1}\sum_{j=1}^{m}\sum_{i=1}^{k-1} r_0\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right) = 0,$$

and

$$mk(1-\rho^2) - \sum_{j=1}^{m}\sum_{i=1}^{k}\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right)^2 + \rho C^*(\theta) + (1-\rho^2)\sum_{j=1}^{m}\sum_{i=1}^{k-1}\frac{(R_{i(j)}-\mu_1)}{\sigma_1}r_0\left(\frac{R_{i(j)}-\mu_1}{\sigma_1}\right) = 0.$$

The roots of the above likelihood equations have no closed form and these equations have to be solved numerically. However, it is not difficult to carry out an numerical computation using mathematical packages, e.g. Maple or Mathematica.

4.1 Fisher Information Matrix

In order to compute the asymptotic relative efficiencies of the MLE estimator of θ based on m independent sequences each with k bivariate records and times (denoted with RCT), with corresponding estimator based on similar mk bivariate records without times (denoted with RC), and the estimator based on a bivariate random sample of size mk (denoted with IID), we derive the Fisher information matrices in these three data bases.

We have $kI_{X,Y}(\theta) = ((I_{ij}));\ i,j = 1\ldots,5$, such that $I_{ij} = I_{ji};\ j \neq i$ and

$$I_{11} = \frac{k}{\sigma_1^2(1-\rho^2)},\quad I_{12} = 0,\quad I_{13} = \frac{-k\rho}{\sigma_1\sigma_2(1-\rho^2)},\quad I_{14} = 0,\quad I_{15} = 0,$$

$$I_{22} = \frac{k(1-\rho^2/2)}{2\sigma_1^4(1-\rho^2)},\quad I_{23} = 0,\quad I_{24} = \frac{-k\rho^2}{4\sigma_1^2\sigma_2^2(1-\rho^2)},\quad I_{25} = \frac{-k\rho}{2\sigma_1^2(1-\rho^2)},$$

$$I_{33} = \frac{k}{\sigma_2^2(1-\rho^2)},\quad I_{34} = 0,\quad I_{35} = 0,$$

$$I_{44} = \frac{k(1-\rho^2/2)}{2\sigma_2^4(1-\rho^2)},\quad I_{45} = \frac{-k\rho}{2\sigma_2^2(1-\rho^2)},$$

and

$$I_{55} = \frac{k(1+\rho^2)}{(1-\rho^2)^2}.$$

We denote some moments of bivariate records from standard bivariate normal distribution as follows

$$\alpha_{i[i]} = \mathrm{E}(R_i^{0,1}R_{[i]}^{0,1}),\ \alpha_{[i]}^{(2)} = \mathrm{E}(R_{[i]}^{0,1})^2 \quad \text{and} \quad \alpha_i^{(2)} = \mathrm{E}(R_i^2),$$

where $(R_i^{0,1}, R_{[i]}^{0,1})$ is the ith bivariate record from standard normal distribution. Suppose the distribution of (X_i, Y_i) is standard bivariate normal with correlation ρ for i=1,2,..., then

$$Y_i = \rho X_i + \epsilon_i, \tag{8}$$

where the X_i and the ϵ_i are mutually independent and ϵ_i are normal distributed with zero mean and variance equal to $1-\rho^2$. So by considering X-record sequence we have for $i \geq 1$

$$R_{[i]}^{0,1} = \rho R_i^{0,1} + \epsilon_{[i]}, \tag{9}$$

where $\epsilon_{[i]}$ denotes the particular ϵ_i associated with $R_i^{0,1}$. Since X_i and ϵ_i are independent so the sequence $R_i^{0,1}$ is independent of $\epsilon_{[i]}$, the later being mutually independent, each with the same distribution as ϵ_i. So we can conclude from (9) that for any finite function $g(.)$

$$\mathrm{E}(R_{[i]}^{0,1} g(R_i^{0,1})) = \rho \mathrm{E}(R_i^{0,1} g(R_i^{0,1})) \quad \text{and}$$

$$\mathrm{E}((R_{[i]}^{0,1})^2 g(R_i^{0,1})) = \rho^2 \mathrm{E}((R_i^{0,1})^2 g(R_i^{0,1})) + (1-\rho^2)\mathrm{E}(g(R_i^{0,1})).$$

Especially we have

$$\mathrm{E}(R_{[i]}^{0,1} R_i^{0,1}) = \alpha_{i[i]} = \rho\alpha_i^{(2)} \quad \text{and} \quad \mathrm{E}[(R_{[i]}^{0,1})^2] = \alpha_{[i]}^{(2)} = \rho^2\alpha_i^{(2)} + 1 - \rho^2.$$

The log-likelihood function of the first k bivariate records is equal to

$$l(\theta; \mathbf{R}(k), \mathbf{C}(k)) = \sum_{i=1}^{k} L^*(\theta; R_i, R_{[i]}) - \sum_{i=1}^{k-1} \log\left(\Phi\left(\frac{R_i - \mu_1}{\sigma_1}\right)\right),$$

where

$$\begin{aligned} L^*(\theta; R_i, R_{[i]}) &= -\frac{1}{2}\log(\sigma_1^2\sigma_2^2(1-\rho)) - \frac{1}{2(1-\rho^2)} \\ &\times \left[\left(\frac{R_i - \mu_1}{\sigma_1}\right)^2 + \left(\frac{R_{[i]} - \mu_2}{\sigma_2}\right)^2\right. \\ &- \left. 2\rho\left(\frac{R_i - \mu_1}{\sigma_1}\right)\left(\frac{R_{[i]} - \mu_2}{\sigma_2}\right)\right]. \end{aligned}$$

Hence, we obtain $I_{(\mathbf{R}(k),\mathbf{C}(k))}(\theta) = ((I'_{ij}));\ i,j = 1\ldots,5$, such that $I'_{ij} = I'_{ji};\ j \neq i$ and

$$\begin{aligned} I'_{11} &= -\sum_{i=1}^{k} \mathrm{E}\left(\frac{\partial^2}{\partial\mu_1^2} L^*(\theta; R_i, R_{[i]})\right) - \frac{1}{\sigma_1^2}\sum_{i=1}^{k-1}\mathrm{E}(r'_0(R_i^{0,1})) \\ &= \frac{1}{\sigma_1^2}\left[\frac{k}{1-\rho^2} + \sum_{i=1}^{k-1}\mathrm{E}(R_i^{0,1} r_0(R_i^{0,1})) - \sum_{i=1}^{k-1}\mathrm{E}(r_0^2(R_i^{0,1}))\right], \end{aligned}$$

$$
\begin{aligned}
I'_{22} &= -\sum_{i=1}^{k} \mathrm{E}\left(\frac{\partial^2}{\partial(\sigma_1^2)^2} L^*(\theta; R_i, R_{[i]})\right) \\
&\quad -\frac{1}{4\sigma_1^4}\mathrm{E}\left[3\sum_{i=1}^{k-1} R_i^{0,1} r_0(R_i^{0,1}) + \sum_{i=1}^{k-1}(R_i^{0,1})^2 r_0'(R_i^{0,1})\right] \\
&= \frac{1}{\sigma_1^4}\left[\frac{-k}{2} + \frac{1}{(1-\rho^2)}\left\{\sum_{i=1}^{k}\alpha_i^{(2)} - \frac{3}{4}\rho\sum_{i=1}^{k}\alpha_{i[i]}\right\}\right. \\
&\quad \left. -\frac{3}{4}\sum_{i=1}^{k-1}\mathrm{E}(R_i^{0,1} r_0(R_i^{0,1})) - \frac{1}{4}\sum_{i=1}^{k-1}\mathrm{E}((R_i^{0,1})^2 r_0'(R_i^{0,1}))\right] \\
&= \frac{1}{\sigma_1^4}\left[\frac{-k}{2} + \frac{1-3/4\rho^2}{(1-\rho^2)}\sum_{i=1}^{k}\alpha_i^{(2)}\right. \\
&\quad -\frac{3}{4}\sum_{i=1}^{k-1}\mathrm{E}(R_i^{0,1} r_0(R_i^{0,1})) + \frac{1}{4}\sum_{i=1}^{k-1}\mathrm{E}((R_i^{0,1})^3 r_0(R_i^{0,1})) \\
&\quad \left. -\frac{1}{4}\sum_{i=1}^{k-1}\mathrm{E}((R_i^{0,1})^2 r_0^2(R_i^{0,1}))\right], \\
I'_{33} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\mu_2^2} L^*(\theta; R_i, R_{[i]})\right) = \frac{k}{\sigma_2^2(1-\rho^2)},
\end{aligned}
$$

and

$$
\begin{aligned}
I'_{44} &= -\sum_{i=1}^{k} \mathrm{E}\left(\frac{\partial^2}{\partial(\sigma_2^2)^2} L^*(\theta; R_i, R_{[i]})\right) \\
&= \frac{1}{\sigma_2^4}\left[\frac{-k}{2} + \frac{1}{(1-\rho^2)}\left\{\sum_{i=1}^{k}\alpha_{[i]}^{(2)} - \frac{3}{4}\rho\sum_{i=1}^{k}\alpha_{i[i]}\right\}\right] \\
&= \frac{1}{\sigma_2^4}\left[\frac{k}{2} + \frac{\rho^2}{4(1-\rho^2)}\sum_{i=1}^{k}\alpha_i^{(2)}\right].
\end{aligned}
$$

Amini and Ahmadi (2007) showed that

$$
I'_{55} = \frac{(1-\rho^2)\sum_{i=1}^{k}\alpha_i^{(2)} + 2k\rho^2}{(1-\rho^2)^2}.
$$

Furthermore, we have

$$
\begin{aligned}
I'_{12} &= -\sum_{i=1}^{k} \mathrm{E}\left(\frac{\partial^2}{\partial\mu_1\partial\sigma_1^2} L^*(\theta; R_i, R_{[i]})\right) \\
&\quad -\frac{1}{2\sigma_1^3}\mathrm{E}\left[\sum_{i=1}^{k-1} r_0(R_i^{0,1}) + \sum_{i=1}^{k-1} R_i^{0,1} r'_0(R_i^{0,1})\right] \\
&= \frac{1}{\sigma_1^3}\left[\frac{1}{1-\rho^2}\left\{\sum_{i=1}^{k}\alpha_i - \frac{\rho}{2}\sum_{i=1}^{k}\alpha_{[i]}\right\}\right. \\
&\quad \left. -\frac{1}{2}\left\{\sum_{i=1}^{k-1}\mathrm{E}(r_0(R_i^{0,1})) + \sum_{i=1}^{k-1}\mathrm{E}(R_i^{0,1} r'_0(R_i^{0,1}))\right\}\right] \\
&= \frac{1}{\sigma_1^3}\left[\frac{1-\rho^2/2}{1-\rho^2}\sum_{i=1}^{k}\alpha_i - \frac{1}{2}\left\{\sum_{i=1}^{k-1}\mathrm{E}(r_0(R_i^{0,1}))\right.\right. \\
&\quad \left.\left. -\sum_{i=1}^{k-1}\mathrm{E}((R_i^{0,1})^2 r_0(R_i^{0,1})) + \sum_{i=1}^{k-1}\mathrm{E}(R_i^{0,1} r_0^2(R_i^{0,1}))\right\}\right], \\
I'_{13} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\mu_1\partial\mu_2}L^*\right) = \frac{-k\rho}{\sigma_1\sigma_2(1-\rho^2)}, \\
I'_{14} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\mu_1\partial\sigma_2^2}L^*\right) = \frac{-\rho^2}{2\sigma_1\sigma_2^2(1-\rho^2)}\sum_{i=1}^{k}\alpha_i, \\
I'_{15} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\mu_1\partial\rho}L^*\right) = \frac{-\rho}{\sigma_1(1-\rho^2)}\sum_{i=1}^{k}\alpha_i, \\
I'_{23} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\sigma_1^2\partial\mu_2}L^*\right) = \frac{-\rho}{2\sigma_1^2\sigma_2(1-\rho^2)}\sum_{i=1}^{k}\alpha_i, \\
I'_{24} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\sigma_1^2\partial\sigma_2^2}L^*\right) = \frac{-\rho^2}{4\sigma_1^2\sigma_2^2(1-\rho^2)}\sum_{i=1}^{k}\alpha_i^{(2)}, \\
I'_{25} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\sigma_1^2\partial\rho}L^*\right) = \frac{-\rho}{2\sigma_1^2(1-\rho^2)}\sum_{i=1}^{k}\alpha_i^{(2)}, \\
I'_{34} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\mu_2\partial\sigma_2^2}L^*\right) = \frac{\rho}{2\sigma_2^3(1-\rho^2)}\sum_{i=1}^{k}\alpha_i, \\
I'_{35} &= -\sum_{i=1}^{k}\mathrm{E}\left(\frac{\partial^2}{\partial\mu_2\partial\rho}L^*\right) = \frac{1}{\sigma_2(1-\rho^2)}\sum_{i=1}^{k}\alpha_i,
\end{aligned}
$$

and

$$I'_{45} = -\sum_{i=1}^{k} \mathrm{E}\left(\frac{\partial^2}{\partial\sigma_2^2\partial\rho}L^*\right) = \frac{\rho}{2\sigma_2^2(1-\rho^2)}\left[\sum_{i=1}^{k}\alpha_i^{(2)} - 2k\right].$$

The Fisher information matrix in the first k bivariate records and record times is equal to $I_{(\mathbf{R}(k),\mathbf{C}(k),\mathbf{\Delta}(k))}(\theta) = ((I''_{ij}))$; $i,j = 1\ldots,5$, such that $I''_{ij} = I''_{ji}$; $j \neq i$. Since the conditional pdf $f_{\mathbf{\Delta}(k)|\mathbf{R}(k),\mathbf{C}(k)}$ depends only on μ_1 and σ_1 and is free of other parameters, we have for $\{(i,j);\ 1 \leq i \leq 5,\ 1 \leq j \leq 5,\ i \leq j\} - \{(1,1),(2,2),(1,2)\}$,

$$\begin{aligned} I''_{ij} &= I'_{ij} - \mathrm{E}\left(\frac{\partial^2}{\partial\theta_i\partial\theta j}\log f_{\mathbf{\Delta}|\mathbf{R},\mathbf{C}}\right) \\ &= I'_{ij}. \end{aligned}$$

Furthermore

$$\begin{aligned} I''_{11} &= \frac{1}{\sigma_1^2}\left[\frac{k}{(1-\rho^2)} - \sum_{i=1}^{k}\mathrm{E}\left(\frac{1-\Phi(R_i^{0,1})}{\Phi(R_i^{0,1})}h'_0(R_i^{0,1})\right)\right] \\ &= \frac{1}{\sigma_1^2}\left[\frac{k}{(1-\rho^2)} + \sum_{i=1}^{k}\mathrm{E}\left(R_i^{0,1}r_0(R_i^{0,1})\right) + \sum_{i=1}^{k}\mathrm{E}\left(r_0(R_i^{0,1})h_0(R_i^{0,1})\right)\right], \\ I''_{22} &= \frac{1}{\sigma_1^4}\left[\frac{-k}{2} + \frac{1}{2(1-\rho^2)}\left\{2\sum_{i=1}^{k}\alpha_i^{(2)} - \frac{3}{2}\rho\sum_{i=1}^{k}\alpha_{i[i]}\right\}\right. \\ &- \frac{3}{4}\sum_{i=1}^{k}\mathrm{E}\left(R_i^{0,1}\frac{1-\Phi(R_i^{0,1})}{\Phi(R_i^{0,1})}h_0(R_i^{0,1})\right) \\ &- \left.\frac{1}{4}\sum_{i=1}^{k}\mathrm{E}\left((R_i^{0,1})^2\frac{1-\Phi(R_i^{0,1})}{\Phi(R_i^{0,1})}h'_0(R_i^{0,1})\right)\right] \\ &= \frac{1}{\sigma_1^4}\left[\frac{-k}{2} + \frac{1-3/4\rho^2}{(1-\rho^2)}\sum_{i=1}^{k}\alpha_i^{(2)} - \frac{3}{4}\sum_{i=1}^{k}\mathrm{E}\left(R_i^{0,1}r_0(R_i^{0,1})\right)\right. \\ &+ \left.\frac{1}{4}\sum_{i=1}^{k}\mathrm{E}\left((R_i^{0,1})^3r_0(R_i^{0,1})\right) + \frac{1}{4}\sum_{i=1}^{k}\mathrm{E}\left((R_i^{0,1})^2r_0(R_i^{0,1})h_0(R_i^{0,1})\right)\right], \end{aligned}$$

and

$$\begin{aligned}
I''_{12} &= I'_{12} - \mathrm{E}\left(\frac{\partial^2}{\partial\mu_1\partial\sigma_1^2}\log f_{\boldsymbol{\Delta}|\mathbf{R},\mathbf{C}}(\delta|\mathbf{r},\mathbf{s})\right) \\
&= I'_{12} - \sum_{i=1}^{k-1}\mathrm{E}\left(\frac{\partial^2}{\partial\mu_1\partial\sigma_1^2}\log\Phi\left(\frac{R_i-\mu_1}{\sigma_1}\right)\right) \\
&- \sum_{i=1}^{k-1}\mathrm{E}\left(\frac{1-\Phi(R_i^{0,1})}{\Phi(R_i^{0,1})}\frac{\partial^2}{\partial\mu_1\partial\sigma_1^2}\log\left(1-\Phi\left(\frac{R_i-\mu_1}{\sigma_1}\right)\right)\right) \\
&= \frac{1}{\sigma_1^3}\left[\frac{1-\rho^2/2}{1-\rho^2}\sum_{i=1}^{k}\alpha_i - \frac{1}{2}\left\{\sum_{i=1}^{k-1}\mathrm{E}(r_0(R_i^{0,1}))\right.\right. \\
&- \left.\left.\sum_{i=1}^{k-1}\mathrm{E}((R_i^{0,1})^2 r_0(R_i^{0,1})) - \sum_{i=1}^{k-1}\mathrm{E}(R_i^{0,1}h_0(R_i^{0,1})r_0(R_i^{0,1}))\right\}\right].
\end{aligned}$$

4.2 Asymptotic relative efficiencies

In this section, we try to find an answer for the question "which sampling scheme (of **A**, **B** or **C**, introduced in Section 1) leads to more precise estimators of population parameters?", for the special case of bivariate normal distribution. We use the asymptotic variances of the ML estimators which are the trace items of the inverse of the Fisher information matrices contained in the three types of data.

Since μ_1 and μ_2 are location parameters, the Fisher information matrices are free of μ_1 and μ_2. It can be seen that $I^{-1}(1,1)$ is multiple of σ_1^2, $I^{-1}(2,2)$ is multiple of σ_1^4, $I^{-1}(3,3)$ is multiple of σ_2^2, $I^{-1}(4,4)$ is multiple of σ_2^4 and $I^{-1}(5,5)$ is free of σ_1 and σ_2, for I be each of the three mentioned information matrices, where $I^{-1}(i,i)$ is the element of the ith row and the ith column of the matrix I^{-1}.

Thus, the asymptotic relative efficiency (denoted by ARE) given below does not depend on μ_i and σ_i, $i=1,2$. Hence, without loss of generality, we take $\sigma_1=\sigma_2=1$ in evaluating $I^{-1}_{(\mathbf{R}(k),\mathbf{C}(k))}(\theta)$, $I^{-1}_{(\mathbf{R}(k),\mathbf{C}(k),\boldsymbol{\Delta}(k))}(\theta)$ and $(kI_{X,Y})^{-1}(\theta)$ using the above formulas. For simplicity, we denote the ML estimate of θ based on $(\mathbf{R}(k),\mathbf{C}(k))$ by $\hat{\theta}_{\mathbf{A}}$, the MLE of θ based on $(\mathbf{R}(k),\mathbf{C}(k),\boldsymbol{\Delta}(k))$ by $\hat{\theta}_{\mathbf{B}}$ and the MLE of θ based on the bivariate random sample of size mk by $\hat{\theta}_{\mathbf{C}}$. Asymptotic efficiency of the MLE using bivariate record values and times with respect to (w.r.t.) that using bivariate random sample of the same size, and ARE of the MLE using bivariate record values and times w.r.t. that using bivariate record values without times are as follows:

$$\mathrm{ARE}(\hat{\theta}_{i,\mathbf{B}};\hat{\theta}_{i,\mathbf{C}})=\frac{(kI_{X,Y})^{-1}(i,i)}{I^{-1}_{(\mathbf{R}(k),\mathbf{C}(k),\mathbf{\Delta}(k))}(i,i)};$$

$$\mathrm{ARE}(\hat{\theta}_{i,\mathbf{B}};\hat{\theta}_{i,\mathbf{A}})=\frac{I^{-1}_{(\mathbf{R}(k),\mathbf{C}(k))}(i,i)}{I^{-1}_{(\mathbf{R}(k),\mathbf{C}(k),\mathbf{\Delta}(k))}(i,i)},\quad \text{for } i=1,\dots,5.$$

Tables 1 to 5 show the values of ARE of $\mu_1,\sigma_1,\mu_2,\sigma_2$ and ρ, respectively. These are similar for negative and positive values of ρ. Since the values variations for Tables 1 and 2 w.r.t. ρ are very negligible, they are shown in 8 decimal places. The other tables' values are shown in 4 decimal places.

Table 1: The values of $\mathrm{ARE}(\hat{\mu}_{1,\mathbf{B}};\hat{\mu}_{1,\mathbf{C}})$ $(\mathrm{ARE}(\hat{\mu}_{1,\mathbf{B}};\hat{\mu}_{1,\mathbf{A}}))$.

	ρ				
k	0.1	0.3	0.5	0.7	0.9
2	0.75230738	0.75230737	0.75230713	0.75230716	0.75230687
	(1.17312564)	(1.17312564)	(1.17312566)	(1.17312565)	(1.17312497)
3	0.57576506	0.57576512	0.57576506	0.57576490	0.57576826
	(1.28298358)	(1.28298355)	(1.28298357)	(1.28298304)	(1.28298400)
4	0.49074455	0.49074457	0.49074461	0.49074453	0.49074460
	(1.43525201)	(1.43525195)	(1.43525236)	(1.43525122)	(1.43525451)
5	0.47198582	0.47198584	0.47198582	0.47198586	0.47198773
	(1.70970389)	(1.70970339)	(1.70970367)	(1.70970404)	(1.70970474)

Table 2: The values of $\mathrm{ARE}(\hat{\sigma}_{1,\mathbf{B}};\hat{\sigma}_{1,\mathbf{C}})$ $(\mathrm{ARE}(\hat{\sigma}_{1,\mathbf{B}};\hat{\sigma}_{1,\mathbf{A}}))$.

	ρ				
k	0.1	0.3	0.5	0.7	0.9
2	1.17304229	1.17304227	1.17304236	1.17304220	1.17304198
	(1.26641666)	(1.26641666)	(1.26641667)	(1.26641663)	(1.26641643)
3	1.13989568	1.13989568	1.13989570	1.13989544	1.13989669
	(1.25907952)	(1.25907947)	(1.25907955)	(1.25907899)	(1.25908029)
4	1.18477633	1.18477632	1.18477626	1.18477616	1.18477632
	(1.31711383)	(1.31711383)	(1.31711405)	(1.31711337)	(1.31711383)
5	1.36443750	1.36443730	1.36443767	1.36443798	1.36444343
	(1.51709264)	(1.51709230)	(1.51709265)	(1.51709311)	(1.51708756)

Table 3: The values of $\mathrm{ARE}(\hat{\mu}_{2,\mathbf{B}}; \hat{\mu}_{2,\mathbf{C}})$ $(\mathrm{ARE}(\hat{\mu}_{2,\mathbf{B}}; \hat{\mu}_{2,\mathbf{A}}))$.

	ρ				
k	0.1	0.3	0.5	0.7	0.9
2	0.8418	0.8338	0.8182	0.7959	0.7680
	(1.0019)	(1.0173)	(1.0471)	(1.0897)	(1.1432)
3	0.6522	0.6453	0.6319	0.6128	0.5890
	(1.0032)	(1.0285)	(1.0776)	(1.1476)	(1.2345)
4	0.5259	0.5228	0.5169	0.5082	0.4971
	(1.0047)	(1.0417)	(1.1146)	(1.2209)	(1.3571)
5	0.4663	0.4571	0.4504	0.4460	0.4439
	(1.0067)	(1.0604)	(1.1693)	(1.3368)	(1.5680)

Table 4: The values of $\mathrm{ARE}(\hat{\sigma}_{2,\mathbf{B}}; \hat{\sigma}_{2,\mathbf{C}})$ $(\mathrm{ARE}(\hat{\sigma}_{2,\mathbf{B}}; \hat{\sigma}_{2,\mathbf{A}}))$.

	ρ				
k	0.1	0.3	0.5	0.7	0.9
2	1.0017	1.0155	1.0432	1.0850	1.1405
	(1.00002)	(1.0019)	(1.0148)	(1.0592)	(1.1699)
3	1.0034	1.0300	1.0776	1.1302	1.1538
	(1.00002)	(1.0019)	(1.0153)	(1.0617)	(1.1721)
4	1.0049	1.0434	1.1136	1.1911	1.2166
	(1.00003)	(1.0023)	(1.0186)	(1.0765)	(1.2137)
5	1.0062	1.0558	1.1530	1.2800	1.3707
	(1.00003)	(1.0032)	(1.0273)	(1.1165)	(1.3408)

Table 5: The values of $\mathrm{ARE}(\hat{\rho}_{\mathbf{B}}; \hat{\rho}_{\mathbf{C}})$ $(\mathrm{ARE}(\hat{\rho}_{\mathbf{B}}; \hat{\rho}_{\mathbf{A}}))$.

	ρ				
k	0.1	0.3	0.5	0.7	0.9
2	1.0937	1.0926	1.0903	1.0869	1.0823
	(1.0012)	(1.0112)	(1.0310)	(1.0605)	(1.0996)
3	1.2055	1.1928	1.1683	1.1332	1.0897
	(1.0014)	(1.0122)	(1.0332)	(1.0631)	(1.1003)
4	1.3234	1.3003	1.2564	1.1958	1.1235
	(1.0018)	(1.0157)	(1.0420)	(1.0784)	(1.1218)
5	1.4442	1.4154	1.3612	1.2873	1.2004
	(1.0027)	(1.0241)	(1.0645)	(1.1195)	(1.1842)

As we can see from the Tables 1 to 5, considering inter-record times along by bivariate records, provides a distinct improve in estimation of θ. Also, plans **A** and **B** lead to more precise estimators of scale and correlation parameters of bivariate normal distribution. On the other hand, location parameters' estimates are more precise based on plan **C**. Furthermore, smaller values of k, for location parameters, and greater values of k, for scale and correlation parameters, lead to improvement of the estimators based on plans **A** and **B** relative to the estimators based on plan **C**. Also, $\mathrm{ARE}(\hat{\sigma}_{2,\mathbf{B}}; \hat{\sigma}_{2,\mathbf{C}})$ is relative to ρ and $\mathrm{ARE}(\hat{\mu}_{2,\mathbf{B}}; \hat{\mu}_{2,\mathbf{C}})$ and $\mathrm{ARE}(\hat{\rho}_{\mathbf{B}}; \hat{\rho}_{\mathbf{C}})$ have reverse relation with ρ.

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