Progressive Type II Censored Order Statistics and their Concomitants: Some Stochastic Comparisons Results

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Abstract. In this paper we prove some stochastic comparisons results for progressive type II censored order statistics. The problem of stochastically comparing concomitants of the two progressive type II censored order statistics with possibly different schemes, under different kinds of dependence between X and Y is considered and it is proved that if Y is stochastically increasing (decreasing) in X, in some senses, then the concomitant variables $Y_{[i:n]}^{\widetilde{R}}$'s are stochastically increasing (decreasing) according to usual stochastic ordering, hazard rate ordering, likelihood ratio ordering, mean residual life ordering and dispersive ordering.

1 Introduction

Let $X_1, \ldots, X_N$ be independent lifetimes of N identical units, with X_i having absolutely continuous distribution function F. These units are placed on test at time $t = 0$. At the time of the ith failure, R_i, $1 \leq i \leq n$, number of surviving units are randomly withdrawn from the experiment. Thus, if n failures are observed then $R_1 + \ldots + R_n$ number of units are progressively censored; hence $N = n + R_1 + \ldots + R_n$. The censoring scheme is denoted by the vector $\widetilde{R} = (R_1, \ldots, R_n)$ and $X_{i:n}^{\widetilde{R},N}$, $i = 1, \ldots, n$, the ith failure time, is called the ith progressive type II censored order statistic.

It is well known that the joint probability density function of $X_{1:n}^{\widetilde{R},N}, \ldots, X_{n:n}^{\widetilde{R},N}$ is given by

$$f_{X_{1:n}^{\widetilde{R},N}, \ldots, X_{n:n}^{\widetilde{R},N}}(x_1, \ldots, x_n) = c \prod_{i=1}^{n} f(x_i)(1 - F(x_i))^{R_i},$$
$$-\infty < x_1 \leq \ldots \leq x_n < \infty$$

where $N = n + \sum_{i=1}^{n} R_i$, $N \in \mathbb{N}$ and $c = \prod_{i=1}^{n} \gamma_i$, $\gamma_i = N - \sum_{j=1}^{i-1} R_j - i + 1$.

In recent year, distribution theory, deriving bounds for the moments and problems of testing and finding application for progressive type II censored order statistics have been study by many researchers including Balakrishnan and Aggarwala (2000), Balakrishnan et al. (2001), Guilbaud (2004) and Alvarez-Andradea et al. (2006).

If $R_1 = \ldots = R_n = 0$ and $n = N$, where no withdrawals are made, the order statistics from progressive type II censoring based on distribution F are reduced to ordinary order statistics based on random sample of size N from distribution F.

If $R_1 = \ldots = R_{n-1} = 0$ and $R_n = N - n$, the order statistics from progressive type II censoring are reduced to usual type II censored order statistics.

Let $(X_1, Y_1), \ldots, (X_N, Y_N)$ be a random sample of size N from a continuous bivariate distribution. Then the Y values associated with $X_{i:n}^{\widetilde{R},N}$ is called the concomitant of the ith progressive type II censored order statistic and denoted by $Y_{[i:n]}^{\widetilde{R},N}$. Let $f(y \mid x)$ denote the conditional probability density function of Y given $X = x$. Then the joint probability density function of the k-concomitants $Y_{[r_1:n]}^{\widetilde{R},N}, \ldots, Y_{[r_k:n]}^{\widetilde{R},N}$

is given by

$$f_{Y_{[r_1:n]}^{\widetilde{R},N},\ldots,Y_{[r_k:n]}^{\widetilde{R},N}}(y_1,\ldots,y_k) =$$

$$\int_{-\infty}^{\infty}\int_{\infty}^{x_k}\cdots\int_{-\infty}^{x_2}\prod_{i=1}^{k}f(y_i\mid x_i)f_{X_{r_1:n}^{\widetilde{R},N},\ldots,X_{r_k:n}^{\widetilde{R},N}}(x_1,\ldots,x_k)\prod_{i=1}^{k}dx_i.$$

From this we obtain the marginal probability density function of the rth concomitant $Y_{[r:n]}^{\widetilde{R},N}$ as

$$f_{Y_{[r:n]}^{\widetilde{R},N}}(y) = \int_{-\infty}^{\infty} f(y\mid x)f_{X_{r:n}^{\widetilde{R},N}}(x)dx. \tag{1.1}$$

Stochastic orderings between concomitants of ordinary order statistics has been discussed in Khaledi and Kochar (2000b). Using their motivation, in this paper we consider the problem of stochastic comparisons among concomitant of order statistics arise from two progressive type II samplings with possibly different numbers of failures and different schemes. Thus, extending some of the results in Khaledi and Kochar (2000b) for the progressive type II censored order statistics with possibly different sample sizes.

The results obtained in this paper are more general and can be applied to concomitants of progressive type II censored order statistics based on any random vector (X, Y) with monotone dependence between the random variables X and Y.

There are several notions of stochastic ordering among random variables with varying degree of strength. In the following, we briefly review some of these notions that will be used later on in this paper.

Notions of Stochastic Orderings

Let X and Y be random variables with distribution functions F and G, survival functions $\bar{F}$ and $\bar{G}$, density functions f and g, and hazard rates r_F $(= f/\bar{F})$ and r_G $(= g/\bar{G})$, respectively. X is said to be stochastically smaller than Y (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x. It is know that

$$X \leq_{st} Y \Leftrightarrow E(\phi(X)) \leq E(\phi(Y)) \text{ for all increasing functions } \phi,$$
$$\tag{1.2}$$

provided that the expectations exist. A stronger notion of stochastic dominance is that of *hazard rate* ordering (denote by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x. In case the hazard rates exist, $X \leq_{hr} Y$ if and only if $r_G(x) \leq r_F(x)$ for every x. X is said to be smaller than Y in *likelihood ratio* ordering (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x. Finally, X is said to be smaller than Y in *mean residual life* (*MRL*) ordering (denoted by $X \leq_{mrl} Y$) if $\int_t^{+\infty} \bar{G}(x)dx / \int_t^{+\infty} \bar{F}(x)dx$ is increasing in t. In this case $\mu_F(x) \leq \mu_G(x)$ for every x, where $\mu_F(x) = E[X - x \mid X > x]$ denoted the mean residual life function of X. Similarly we defined $\mu_G(x)$. When the supports of X and Y have a common left end-point, we have the following chain of implications among the above stochastic orders: $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$. Also $X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$. For more details on stochastic ordering, see Shaked and Shanthikumar (1994).

There are several notions of positive and negative dependence between random variables and these have been discussed in detail in Lehmann (1966), Barlow and Proschan (1981), Shaked (1977) and Lee (1985a, b). For a brief introduction, see Boland et al. (1996). We use the following concepts in this paper.

Definition 1.1. *We say that a function $h(x, y)$ is* sign regular *of order 2 (SR_2) if $\epsilon_1 h(x, y) \geq 0$ and*

$$\epsilon_2(h(x_1, y_1)h(x_2, y_2) - h(x_2, y_1)h(x_1, y_2)) \geq 0$$

whenever $x_1 < x_2, y_1 < y_2$, and $\epsilon_i \in \{-1, 1\}$ for $i = 1, 2$.

If the above inequalities hold with $\epsilon_1 = +1$ and $\epsilon_2 = +1$ then h is said to be totally positive of order 2 (TP_2); and if they hold with $\epsilon_1 = +1$ and $\epsilon_2 = -1$ then h is said to be reverse regular of order 2 (RR_2).

Let X and Y be random variables with joint distribution function F and density f. For $s > 0$, let $\gamma^{(s)}(t)$ be defined as follows:

$$\gamma^{(s)}(t) = \begin{cases} (-t)^{s-1}/\Gamma(s) & \text{if} \quad t \leq 0 \\ 0 & \text{if} \quad t > 0, \end{cases}$$

where $\Gamma(.)$ is the complete gamma function.
Define the 2-fold integral ψ_{k_1, k_2} by

$$\psi_{k_1, k_2}(x_1, x_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \gamma^{(k_1)}(x_1 - t_1)\gamma^{(k_2)}(x_2 - t_2)dF(t_1, t_2)$$

and define $\psi_{0,0}(x,y) = f(x,y)$. Also define

$$\psi_{0,k_2}(x,y) = \int_{-\infty}^{+\infty} \gamma^{(k_2)}(y-t_2)g_1(x)dF(t_2 \mid x)$$

where g_1 is the density of X and $F(t_2 \mid x)$ is the conditional distribution function of Y given $X = x$, for $k_2 > 0$. Similarly we can define $\psi_{k_1,0}(x,y)$ for $k_1 > 0$. Shaked (1977) introduced the following concept of positive dependence for the bivariate vector (X,Y).

Definition 1.2. *The random vector (X,Y) is said to be dependent by total positivity with degree (k_1, k_2), denoted by $DTP(k_1, k_2)$, if $\psi_{k_1,k_2}(x,y)$ is TP_2.*

The corresponding concept of negative dependence was introduced by Lee (1985a, b).

Definition 1.3. *We say that (X,Y) is dependent by reverse regular of degree (k_1, k_2), denoted by $DRR(k_1, k_2)$, if $\psi_{k_1,k_2}(x,y)$ is RR_2.*

As pointed out by Shaked (1977), two random variables X and Y are likelihood ratio (or TP_2) dependent if and only if X and Y are $DTP(0,0)$ dependent. They are $DTP(0,1)$ ($DRR(0,1)$) dependent if the conditional hazard rate of Y given $X = x$, $r(y|x)$, is decreasing (increasing) in x. The random variables X and Y are $DTP(1,1)$ dependent if the joint survival function $\bar{F}(x,y) = P[X > x, Y > y]$ of (X,Y) is TP_2. In this case the random variables X and Y are also said to be right corner set increasing ($RCSI$). The random variables X and Y are $DTP(0,2)$ ($DRR(0,2)$) dependent on whether the conditional mean residual life function of Y given $X = x$, $\mu(y|X = x)$, is increasing (decreasing) in x. We say that Y is stochastically increasing (decreasing) in X, denoted by $SI(Y|X)$ ($SD(Y|X)$), if $P[Y > y|X = x]$ is increasing (decreasing) in x for all y. There are many other notions of positive and negative dependence, but we will not be discussing them here. See Karlin and Rinott (1980a, b) for many interesting examples of bivariate distributions which satisfy the above criteria of dependence.

In Section 2, we prove some stochastic comparisons results for progressive type II censored order statistics. The problem of stochastically comparing $Y_{[i:n]}^{\widetilde{R},N}$ with $Y_{[i:m]}^{\widetilde{R}^*,N^*}$ under different kinds of dependence between X and Y, where $\widetilde{R} = (R_1, \ldots, R_n)$ and $\widetilde{R}^* = (R_1^*, \ldots, R_m^*)$ are possibly different schemes is considered in Section 3.

2   Stochastic orderings among progressive type II censored order statistics

In this section we turn our attention to stochastic comparison of progressive type II censored order statistics from two possible different censoring schemes. We will use the following known results to prove the main results in this section. They can be found in Shaked and Shanthikumar (1994).

Theorem 2.1.   *Let E_γ denote an exponential random variable with hazard rate $\gamma > 0$, Then, $\gamma \geq \gamma'$ implies*

$$E_\gamma \leq_{lr} E_{\gamma'}.$$

Theorem 2.2.   *Let $\{X_i\}$ and $\{Y_i\}$ be two sequences of independent nonnegative random variables. If, for $i \geq 1$, $X_i \leq_{lr} Y_i$ and either X_i or Y_i has logconcave density function, then,*

$$\sum_{i=1}^{m} X_i \leq_{lr} \sum_{i=1}^{n} Y_i \quad for \quad n \geq m.$$

Theorem 2.3.   *Let $X \leq_{lr} Y$ and g be an increasing function. Then, $g(X) \leq_{lr} g(Y)$.*

In the next theorem we prove likelihood ratio ordering between progressive type II censored order statistics which is an extension of Theorem 1.2 of Korwar (2003).

Theorem 2.4.   *Let $X_{1:n}^{\widetilde{R},N}, \ldots, X_{n:n}^{\widetilde{R},N}$ and $X_{1:m}^{\widetilde{R}^*,N^*}, \ldots, X_{m:m}^{\widetilde{R}^*,N^*}$ be two sets of the progressive type II censored order statistics of sizes n and m based on distribution F with parameters $\gamma_k = n - k + 1 + \sum_{l=k}^{n} R_l$ and $\gamma_k^* = m - k + 1 + \sum_{l=k}^{m} R_l^*$, respectively. Then for $i \leq j$,*

$$\gamma_k \geq \gamma_{l_k}^*, \text{ for some set } \{l_1, \ldots, l_i\} \subset \{1, \ldots, j\}, \qquad (2.1)$$

implies that

$$X_{i:n}^{\widetilde{R},N} \leq_{lr} X_{j:m}^{\widetilde{R}^*,N^*}.$$

Proof. Let $E_{i:n}^{\widetilde{R},N}$ be the ith progressive type II censored order statistic based on standard exponential distribution $F_E(x) = 1 - e^x$ and let $E_{\gamma_1}, \ldots, E_{\gamma_n}$ be independent exponential random variables with

E_{γ_i} having hazard rate γ_i, $i = 1, \ldots, n$. Then, it is shown in Cramer and Kamps (2003) that

$$E_{i:n}^{\widetilde{R},N} =_{st} \sum_{k=1}^{i} E_{\gamma_k} \text{ and } X_{i:n}^{\widetilde{R}:N} =_{st} F^{-1} \mathrm{o} F_E(\sum_{k=1}^{i} E_{\gamma_k}), \qquad (2.2)$$

where st means equal in distribution and F^{-1} is the right continuous inverses of the F. Using assumption (2.1), it follows from Theorem 2.1. and 2.2. that

$$\begin{aligned} \sum_{k=1}^{i} E_{\gamma_k} \quad &\leq_{lr} \quad \sum_{k=1}^{i} E_{\gamma_{l_k}^*} \\ &\leq_{lr} \quad \sum_{k=1}^{i} E_{\gamma_{l_k}^*} + \sum_{k \notin \{l_1,\ldots,l_i\}} E_{\gamma_{l_k}^*} \\ &\leq_{lr} \quad \sum_{k=1}^{j} E_{\gamma_k^*} \qquad (2.3) \end{aligned}$$

Now, it follows from Theorem 2.3. that

$$X_{i:n}^{\widetilde{R},N} = F^{-1} \mathrm{o} F_E(\sum_{k=1}^{i} E_{\gamma_k}) \leq_{lr} F^{-1} \mathrm{o} F_E(\sum_{k=1}^{j} E_{\gamma_k^*}) = X_{j:m}^{\widetilde{R}^*,N^*},$$

since $F^{-1} \mathrm{o} F_E(x)$ is an increasing function of x. This proves the required result. ∎

Corollary 2.1. *Let $X_{1:n}^{\widetilde{R},N}, \ldots, X_{n:n}^{\widetilde{R},N}$ be a set of the progressive type II censored order statistics of sizes n based on distribution F. Then*

$$X_{i:n}^{\widetilde{R},N} \leq_{lr} X_{i+1:n}^{\widetilde{R},N}.$$

A random variable X with distribution function F is said to be more dispersed than another variable Y with distribution function G, written as $X \leq_{disp} Y$ or $F \leq_{disp} G$, if and only if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha < \beta < 1$, where G^{-1} is the right continuous inverse of the G. It is known that

$$F \leq_{disp} G \Leftrightarrow G^{-1}F(x) - x \text{ is increasing in } x. \qquad (2.4)$$

For general information about dispersive ordering and its properties, the reader is referred to Shaked and Shanthikumar (1994, Section 2.B).

Next, we prove dispersive ordering between two progressive type II censored order statistics from two possible different censoring schemes. The proof of the next lemma is omitted. Since it is similar to that of lemma 2.1 in Khaledi and Kochar (2000a).

Lemma 2.1. *Let $E_{i:n}^{\widetilde{R},N}$ and $E_{j:m}^{\widetilde{R}^*,N^*}$ be the ith and jth progressive type II censored order statistics based on standard exponential distribution. If (2.1) holds, then for $i \leq j$,*

$$E_{i:n}^{\widetilde{R},N} \leq_{disp} E_{j:m}^{\widetilde{R}^*,N^*}.$$

We shall need the following result due to Bartoszewicz (1987) to extend the above result from exponential distribution to DFR distribution.

Lemma 2.2. *Let $\phi : \Re_+ \to \Re_+$ be a function such that $\phi(0) = 0$ and $\phi(x) - x$ is increasing. Then for every convex and strictly function $\psi : \Re_+ \to \Re_+$ the function $\psi \phi \psi^{-1}(x) - x$ is increasing.*

Theorem 2.5. *Let $X_{1:n}^{\widetilde{R},N}, \ldots, X_{n:n}^{\widetilde{R},N}$ and $X_{1:m}^{\widetilde{R}^*,N^*}, \ldots, X_{m:m}^{\widetilde{R}^*,N^*}$ be two sets of the progressive type II censored order statistics of sizes n and m based on DFR distribution F with parameters $\gamma_k = n - k + 1 + \sum_{l=k}^{n} R_l$ and $\gamma_k^* = m - k + 1 + \sum_{l=k}^{m} R_l^*$, respectively. If condition (2.1) holds, then for $i \leq j$,*

$$X_{i:n}^{\widetilde{R},N} \leq_{disp} X_{j:m}^{\widetilde{R}^*,N^*}.$$

Proof. Let $F_{i:n}(x)$ and $F_{j:m}^*(x)$ denote the distribution functions of $X_{i:n}^{\widetilde{R},N}$ and $X_{j:m}^{\widetilde{R}^*,N^*}$, respectively. Also, let $G_{i:n}$ and $G_{j:m}^*$ denote the distribution functions of $E_{i:n}^{\widetilde{R},N}$ and $E_{j:m}^{\widetilde{R}^*,N^*}$, respectively. To prove the required result we have to show that

$$F_{j:m}^{*-1} F_{i:n}(x) - x \quad \uparrow x. \tag{2.5}$$

Using (2.2), (2.5) is equivalent to

$$F^{-1} o F_E o G_{j:m}^{*-1} o G_{i:n} o F_E^{-1} o F(x) - x \quad \uparrow x.$$

By Lemma 2.1. , $G_{j:m}^{*-1} o G_{i:n}(x) - x$ is increasing in x. Also the function $\psi(x) = F^{-1} o F_E(x)$ is strictly increasing and convex if F is DFR. Now, the required result follows from Lemma 2.2. ∎

3   Stochastic ordering among concomitants of progressive type II censored order statistics

We shall be using the following theorem to prove the main results in this section.

Theorem 3.1.   *(Shaked and Shanthikumar, 1994) Consider a family of distribution $\{G_\theta, \ \theta \in \chi\}$ and let $X(\theta)$ denotes a random variable with distribution function G_θ. Let Θ_1 and Θ_2 be two random variables with supports in χ and distribution functions F_1 and F_2, respectively, such that*

$$\Theta_1 \leq_{lr} \Theta_2. \tag{3.1}$$

Let Y_1 and Y_2 be two random variables such that $Y_i =_{st} X(\Theta_i)$, $i = 1, 2$, that is, suppose that the probability density function of Y_i is given by

$$h_i(y) = \int_\chi g_\theta(y) dF_i(\theta), \quad y \in \Re, \quad i = 1, 2.$$

Then, for $\theta \leq \theta'$

(a) $X(\theta) \leq_{lr} X(\theta') \Rightarrow Y_1 \leq_{lr} Y_2$,

(b) $X(\theta) \leq_{hr} X(\theta') \Rightarrow Y_1 \leq_{hr} Y_2$,

(c) $X(\theta) \leq_{rh} X(\theta') \Rightarrow Y_1 \leq_{rh} Y_2$,

(d) $X(\theta) \leq_{st} X(\theta') \Rightarrow Y_1 \leq_{st} Y_2$ *and*

(e) $X(\theta) \leq_{mrl} X(\theta') \Rightarrow Y_1 \leq_{mrl} Y_2$.

Theorem 3.2.   *Under the assumptions of Theorem 2.4, for $i \leq j$ we have*

(a) $f(x,y)$ *is* TP_2 *in* $(x,y) \ \Rightarrow Y_{[i:n]}^{\widetilde{R},N} \leq_{lr} Y_{[j:m]}^{\widetilde{R}^*,N^*}$,

(b) $r(y|x) \downarrow x \ \Rightarrow Y_{[i:n]}^{\widetilde{R},N} \leq_{hr} Y_{[j:m]}^{\widetilde{R}^*,N^*}$,

(c) $\tilde{r}(y|x) \uparrow x \Rightarrow Y_{[i:n]}^{\widetilde{R},N} \leq_{rh} Y_{[j:m]}^{\widetilde{R}^*,N^*}$,

(d) $SI(Y \mid X) \ \Rightarrow Y_{[i:n]}^{\widetilde{R},N} \leq_{st} Y_{[j:m]}^{\widetilde{R}^*,N^*}$,

(e) $E(Y|X = x) \uparrow x \Rightarrow Y_{[i:n]}^{\widetilde{R},N} \leq_{mrl} Y_{[j:m]}^{\widetilde{R}^*,N^*}$,

where $r(y|x)$ and $\tilde{r}(y|x)$ are respectively, the hazard rate and the reverse hazard rate of the conditional distribution of Y given $X = x$.

Proof. The probability density functions of the $Y_{[i:n]}^{\widetilde{R},N}$ and $Y_{[j:m]}^{\widetilde{R}^*,N^*}$ are, respectively,

$$g_{Y_{[i:n]}^{\widetilde{R},N}}(y) = \int_{-\infty}^{+\infty} f(y|x) f_{i:n}(x) dx$$

and

$$g_{Y_{[j:m]}^{*\widetilde{R}^*,N^*}}^*(y) = \int_{-\infty}^{+\infty} f(y|x) f_{j:m}^*(x) dx.$$

It follows from Theorem 2.4. that

$$X_{i:n}^{\widetilde{R},N} \leq_{lr} X_{j:m}^{\widetilde{R}^*,N^*}.$$

On the other hand,

$$(Y|X = x) \quad =_{st} \quad (Y_{[i:n]}^{\widetilde{R},N} | X_{i:n}^{\widetilde{R},N} = x) \tag{3.2}$$

$$=_{st} \quad (Y_{[j:m]}^{\widetilde{R}^*,N^*} | X_{j:m}^{\widetilde{R}^*,N^*} = x). \tag{3.3}$$

Now, using the above observations, the required results of parts a, b, c, d and e follow from the corresponding parts in Theorem 3.1. $\blacksquare$

The following result which is a special case of Theorem 3.2. , is of great interest.

Corollary 3.1. *Let $X_{i:n}^R$ denotes the ith progressive type II censored order statistics based on distribution F with equal withdrawal R. Then for $i \leq j$,*

(a) $f(x,y)$ is TP_2 in $(x,y) \Rightarrow Y_{[i:n]}^R \leq_{lr} Y_{[j:n]}^R$,

(b) $r(y|x) \downarrow x \Rightarrow Y_{[i:n]}^R \leq_{hr} Y_{[j:n]}^R$,

(c) $\tilde{r}(y|x) \uparrow x \Rightarrow Y_{[i:n]}^R \leq_{rh} Y_{[j:n]}^R$,

(d) $SI(Y \mid X) \Rightarrow Y_{[i:n]}^R \leq_{st} Y_{[j:n]}^R$,

(e) $E(Y|X = x) \uparrow x \Rightarrow Y_{[i:n]}^R \leq_{mrl} Y_{[j:n]}^R$.

Next result is devoted to stochastic comparisons of concomitants of order statistics which are extensions of the results in Khaledi and Kochar (2000b) from equal sample sizes to possible different sample sizes.

Corollary 3.2. *Let $X_{i:n}$ $(X_{j:m})$ be the ith (jth) order statistic of a random sample of size n (m) from a distribution F. Then for $i \leq j$ and $n - i \geq m - j$,*

(a) $f(x,y)$ is TP_2 in (x,y) $\Rightarrow Y_{[i:n]} \leq_{lr} Y_{[j:m]}$,

(b) $r(y|x) \downarrow x \Rightarrow Y_{[i:n]} \leq_{hr} Y_{[j:m]}$,

(c) $\tilde{r}(y|x) \uparrow x \Rightarrow Y_{[i:n]} \leq_{rh} Y_{[j:m]}$,

(d) $SI(Y \mid X) \Rightarrow Y_{[i:n]} \leq_{st} Y_{[j:m]}$,

(e) $E(Y|X=x) \uparrow x \Rightarrow Y_{[i:n]} \leq_{mrl} Y_{[j:m]}$

Remark 1. *The inequalities results obtained in Theorem 3.1., 3.2. and Corollary 3.1. and 3.2. are reversed if we replace the positive dependence condition between X and Y with the corresponding negative dependence condition.*

Now, we prove dispersive ordering between concomitants of progressive type II censored order statistics. To do so, we need the following results due to Bagai and Kochar (1986).

Theorem 3.3. *Let X and Y be two random variables such that they have a common left end points of their supports. If X or Y is DFR and $X \leq_{hr} Y$, then $X \leq_{disp} Y$.*

Theorem 3.4. *Let $r(y|x)$ be decreasing in x and y and the left endpoint of the support of the distribution of $(Y|X = x)$ does not depend on x. Then, for $i \leq j$ and $\gamma_{l_k} \geq \gamma_{l_k}^*$ for some set $\{l_1, \ldots, l_i\} \subset \{1, \ldots, j\}$,*

$$Y_{[i:n]}^{\widetilde{R},N} \leq_{disp} Y_{[j:m]}^{\widetilde{R}^*,N^*}.$$

Proof. It is known that a mixture of DFR distributions is DFR distribution (cf. Barlow and Proschan, 1981). By assumption, $r(y|x)$ is a decreasing function of y for each x. Thus, the distribution of

$Y_{[i:n]}^{\widetilde{R},N}$ is a mixture of DFR distribution, and hence is DFR. On the other hand, it follows from Theorem 3.2. (b) that

$$Y_{[i:n]}^{\widetilde{R},N} \leq_{hr} Y_{[j:m]}^{\widetilde{R}^*,N^*}.$$

Combining these observations, the required result follows from Theorem 3.3. ∎

This result generalizes Theorem 3.7 in Khaledi and Kochar (2000b) from ordinary order statistics to progressive type II censored order statistics, for possible different sample sizes.

Remark 2. *These results may have potential applications in the study of small sample properties of various estimates and tests for independence based on concomitants of progressive type II censored order statistics.*

Remark 3. *It is worth to mention that all the results for concomitants of progressive type II censored order statistics are also valid for concomitants of generalized order statistics.*

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References

Alvarez-Andradea, S., Balakrishnan, N., and Bordesa, L. (2006), Homogeneity tests based on several progressively type II censored samples. J. Multivariate Anal., To appear.

Bagai, I. and Kochar, S. (1986), On tail-ordering and comparison of failure rates. Comm. Statist., Theory Methods, **15**, 1377-1388.

Balakrishnan, N. and Aggarwala, A. (2000), Progressive Censoring-Theory, Methods, and Application. Boston: Birkhäuser.

Balakrishnan, N., Cramer, E., and Kamps, U. (2001), Bounds for means and variances of progressive type II censored order statistics. Statist. Probab. Lett., **54**, 301-315.

Barlow, E. and Proschan, F. (1981), Statistical Theory of Reliability and Life Testing. To Begin With, Silver Spring, MD.

Bartoszewicz, J. (1987), A note on dispersive ordering defined by hazard functions. Statist. Probab. Lett., **6**, 13-16.

Boland, P. J., Hollander, M., Joag-Dev, K., and Kochar, S. (1996), Bivariate dependence properties of order statistics. J. Multivariate Anal., **56**, 75-89.

Cramer, E. and Kamps, U. (2003), Marginal distributions of sequential and generalized order statistics. Metrika, **58**, 293-310.

Guilbaud, O. (2004), Exact non-parametric confidence, prediction and tolerance intervals with progressive type II censoring. Scand. J. Statist., **31**, 265-281.

Karlin, S. and Rinoot, Y. (1980a), Classes of ordering of measures and related correlation inequalities, I: Multivariate totally positive distributions. J. Multivariate Anal., **10**, 467-498.

Karlin, S. and Rinoot, Y. (1980b), Classes of ordering of measures and related correlation inequalities, II: Multivariate reverse rule distributions. J. Multivariate Anal. **10**, 499-516.

Khaledi, B. E. and Kochar, S. (2000a), On despersive ordering between order statistics in one-sample and two-sample problems. Statist. Probab. Lett., **46**, 257-261.

Khaledi, B. E. and Kochar, S. (2000b), Stochastic comparison and dependence among to concomitants of order statistics. J. Multivariate Anal., **73**, 262-281.

Korwar, R. (2003), On the likelihood ratio for progressive type II censored order statistics. Sankhyā., **65**, 793-798.

Lee, M. L. T. (1985a), Dependence by total positivity. Ann. Probab., **13**, 572-582.

Lee, M. L. T. (1985b), Dependence by reverse regular rule. Ann. Probab **13**, 583-591.

Lehmann, E. L. (1966), Some concepts of dependence. Ann. Math. Statist., **37**, 1137-1153.

Shaked, M. and Shanthikumar, J. G. (1994), Stochastic Orders and Their Applications. San Diego, CA: Academic Press.

Shaked, M. (1977), A family of concepts of dependence for bivariate distribution. J. Amer. Statist. Assoc., **72**, 642-650.