JIRSS (2007)
Vol. 6, No. 1, pp 61-75

Multivariate Dispersive Ordering of Generalized Order Statistics

Jing Chen[1]**, Taizhong Hu**[2]

[1]Business School, Shenzhen University, Shenzhen, Guangdong 518060, People's Republic of China.
[2]Department of Statistics and Finance, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China.
(thu@ustc.edu.cn)

Abstract. The concept of generalized order statistics (GOSs) was introduced as a unified approach to a variety of models of ordered random variables. The purpose of this paper is to investigate conditions on the underlying distribution functions and the parameters on which GOSs are based, to establish Shaked-Shanthikumar multivariate dispersive ordering of GOSs from one sample and Khaledi-Kochar multivariate dispersive ordering of GOSs from two samples. Some applications are also given.

1 Introduction

Let X and Y be two random variables with distribution functions F and G, respectively. X is said to be less dispersed than Y, denoted

Key words and phrases: Copula, DFR, directionally convex function, dispersive order, hazard rate order, nonhomogeneous Poisson process, record values.

by $X \leq_{\text{disp}} Y$ or $F \leq_{\text{disp}} G$, if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \text{ whenever } 0 < \alpha < \beta < 1,$$

where F^{-1} and G^{-1} are the right continuous inverses of F and G, respectively. The univariate dispersive order has been studied extensively (see Shaked and Shanthikumar, 1994). In the past ten years, several attempts have been made to extend the dispersive order from the univariate to the multivariate.

Giovagnoli and Wynn (1995) gave a definition of the multivariate dispersive order by means of a contraction function between two random vectors. A function $\mathbf{K} : \Re^n \to \Re^n$ is said to be a contraction function if

$$||\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}')||_2 \leq ||\mathbf{x} - \mathbf{x}'||_2 \text{ for all } \mathbf{x}, \mathbf{x}' \in \Re^n,$$

where $||\cdot||_2$ is the Euclidean norm. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors. $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in *Giovagnoli-Wynn multivariate dispersive order* if there exists a contraction function $\mathbf{K}(\cdot)$ such that $\mathbf{X} \stackrel{\text{st}}{=} \mathbf{K}(\mathbf{Y})$. Fernádez-Ponse and Rodríguez-Griñolo (2006) investigated sufficient and necessary conditions under which Giovagnoli-Wynn multivariate dispersive order is preserved through properties of the corresponding transformation, and establish such a ordering for Wishart distributions.

Considering the difficulty in determining the above contraction function $\mathbf{K}$, Fernádez-Ponse and Suárez-Lloréns (2003) defined a multivariate dispersive order through conditional quantiles which are more widely separated. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector with joint distribution function F. Denote by F_1 the marginal distribution function of X_1, and denote by $F_{i|1,\ldots,i-1}(\cdot|x_1, \ldots, x_{i-1})$ the conditional distribution function of X_i given that $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}$ for $i = 2, \ldots, n$. For each $\mathbf{u} = (u_1, \ldots, u_n) \in (0,1)^n$, define

$$x_1(\mathbf{u}) = F_1^{-1}(u_1) \tag{1.1}$$

and, by induction,

$$x_i(\mathbf{u}) = F^{-1}_{i|1,\ldots,i-1}(u_i|x_1, \ldots, x_{i-1}), \quad i = 2, \ldots, n. \tag{1.2}$$

Similarly, for another random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ with joint distribution function G, define

$$y_1(\mathbf{u}) = G_1^{-1}(u_1), \tag{1.3}$$

and, by induction,

$$y_i(\mathbf{u}) = G^{-1}_{i|1,\ldots,i-1}(u_i|y_1, \ldots, y_{i-1}), \quad i = 2, \ldots, n. \tag{1.4}$$

$\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in *Fernádez-Suárez multivariate dispersive order* if

$$||\mathbf{x}(\mathbf{u}) - \mathbf{x}(\mathbf{u}')||_2 \leq ||\mathbf{y}(\mathbf{u}) - \mathbf{y}(\mathbf{u}')||_2, \quad \forall\ \mathbf{u}, \mathbf{u}' \in (0,1)^n,$$

where

$$\mathbf{x}(\mathbf{u}) = (x_1(\mathbf{u}), x_2(\mathbf{u}), \ldots, x_n(\mathbf{u}))$$

and

$$\mathbf{y}(\mathbf{u}) = (y_1(\mathbf{u}), y_2(\mathbf{u}), \ldots, y_n(\mathbf{u})).$$

Arias-Nicolás et al. (2005) established Fernádez-Suárez multivariate dispersive ordering for two multivariate t-distributions.

Based on the monotonicity of two conditional quantiles, Shaked and Shanthikumar (1998) introduced the following multivariate dispersive order.

Definition 1.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* n*-dimensional random vectors. Then* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in Shaked-Shanthikumar multivariate dispersive order, denoted by* $\mathbf{X} \leq_{\text{disp}} \mathbf{Y}$*, if* $y_i(\mathbf{u}) - x_i(\mathbf{u})$ *is increasing in* $(u_1, \ldots, u_i) \in (0,1)^i$ *for* $i = 1, \ldots, n$*, where* $y_i(\mathbf{u})$ *and* $x_i(\mathbf{u})$ *are defined in* (1.1)-(1.4).

Belzunce and Ruiz (2002), and Belzunce et al. (2003) established Shaked-Shanthikumar multivariate dispersive ordering for ordinary order statistics from two samples and epoch times of two nonhomogeneous Poisson processes, respectively. More generally, Belzunce et al. (2005) obtained that

Theorem 1.1. *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ *and* $\{Y_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ *be generalized order statistics* (GOS*s) based on continuous distribution functions* F *and* G*, respectively (The formal definition of* GOS*s is given in Section* 2*). If* $F \leq_{\text{disp}} G$*, then*

$$\begin{aligned}&\left(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right)\\ &\qquad \leq_{\text{disp}} \quad \left(Y_{(1,n,\tilde{m}_n,k)}, Y_{(2,n,\tilde{m}_n,k)}, \ldots, Y_{(n,n,\tilde{m}_n,k)}\right).\end{aligned}$$

Khaledi and Kochar (2005) introduced another new multivariate dispersive order called *upper orthant dispersive order*.

Definition 1.2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* n*-dimensional random vectors. Then* $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the upper orthant dispersive order, denoted by* $\mathbf{X} \leq_{\text{uo-disp}} \mathbf{Y}$*, if for all* $\mathbf{u} \in (0,1)^n$ *and each* $i = 1, \ldots, n$*,*

$$\left[X_i \,\middle|\, \bigcap_{j \neq i} \{X_j > F_j^{-1}(u_j)\} \right] \leq_{\text{disp}} \left[Y_i \,\middle|\, \bigcap_{j \neq i} \{Y_j > G_j^{-1}(u_j)\} \right].$$

Some interesting properties of the multivariate orders $\leq_{\text{uo-disp}}$ and $\leq_{\text{disp}}$ are recalled in Propositions 3.1 and 3.2, and Lemma 2.1 below.

The purpose of this paper is to investigate conditions on the underlying distribution functions and the parameters on which GOSs are based, to establish Shaked-Shanthikumar multivariate dispersive ordering of GOSs from one sample and Khaledi-Kochar multivariate dispersive ordering of GOSs from two samples. The main results are given in Section 2. From these results in Section 2 and properties of the multivariate dispersive orderings, we can obtain many interesting probability inequalities concerning GOSs from one and two samples. Such applications are presented in Section 3.

Throughout, for any distribution function F, $\overline{F} = 1 - F$ denotes its survival function. When an expectation or a probability is conditioned on an event such as $\mathbf{X} = \mathbf{x}$, we assume that $\mathbf{x}$ is in the support of $\mathbf{X}$. All expectations are implicitly assumed to exist wherever they are given. Also, we denote by $[X|A]$ any random variable whose distribution is the conditional distribution of X given event A.

2 Main Results

To state the main results of this section, we recall the definition of GOSs, which was introduced by Kamps (1995a,b) as a unified approach to a variety of models of ordered random variables.

Definition 2.1. [Kamps, 1995a] *Let* $n \in \mathbb{N}$*,* $k \geq 0$*,* $m_1, \ldots, m_{n-1} \in \Re$ *be parameters such that*

$$\gamma_{r,n} = k + \sum_{j=r}^{n-1} (m_j + 1) > 0, \quad r = 1, \ldots, n, \tag{2.1}$$

and let $\widetilde{m}_n = (m_1, \ldots, m_{n-1})$ *if* $n \geq 2$ *(*$\tilde{m}_n$ *arbitrary if* $n = 1$*). If the random variables* $U_{(r,n,\tilde{m}_n,k)}$*,* $r = 1, \ldots, n$*, possess a joint density of* *the form*

$$f_{U_{(1,n,\tilde{m}_n,k)},\dots,U_{(n,n,\tilde{m}_n,k)}}(u_1,\dots,u_n) = k\left(\prod_{j=1}^{n-1}\gamma_{j,n}\right)\left(\prod_{i=1}^{n-1}(1-u_i)^{m_i}\right)(1-u_n)^{k-1}$$

on the cone $0 \le u_1 \le u_2 \le \cdots \le u_n < 1$ *of* $\Re^n$*, then they are called uniform* GOS*s. Now, let* F *be an arbitrary distribution function. The random variables*

$$X_{(r,n,\tilde{m}_n,k)} = F^{-1}(U_{(r,n,\tilde{m}_n,k)}), \quad r = 1,\dots,n, \tag{2.2}$$

are called the GOS*s based on* F*.*

Ordinary order statistics of a random sample from distribution F are a particular case of GOSs when $k = 1$ and $\widetilde{m}_n = (0,\dots,0)$. When $k = 1$ and $\widetilde{m}_n = (-1,\dots,-1)$, we then get the first n record values from a sequence of random variables with distribution F. Choosing the parameters appropriately, several other models of ordered random variables are seen to be particular cases.

It is well known that GOSs from a continuous distribution form a Markov chain with transition probabilities

$$\mathbb{P}[X_{(r,n,\tilde{m}_n,k)} > t | X_{(r-1,n,\tilde{m}_n,k)} = s] = \left[\frac{\overline{F}(t)}{\overline{F}(s)}\right]^{\gamma_{r,n}} \tag{2.3}$$
$$\text{for } t \ge s \text{ and } r = 2,\dots,n.$$

For any two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$, denote by F_i and G_i the marginal distribution functions of X_i and Y_i for each i, respectively, and denote by

$$F_{i|1,\dots,i-1}(\cdot|x_1,\dots,x_{i-1}) \quad \text{and} \quad G_{i|1,\dots,i-1}(\cdot|y_1,\dots,y_{i-1})$$

the conditional distribution functions of $[X_i|X_1 = x_1,\dots,X_{i-1} = x_{i-1}]$ and $[Y_i|Y_1 = y_1,\dots,Y_{i-1} = y_{i-1}]$, respectively, for $i \ge 2$. Next, recall that a nonnegative random variable X with distribution function F is said to have *decreasing failure rate* (DFR) if $\overline{F}(x)$ is log-convex in $x \in \Re_+$.

Theorem 2.1. *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r = 1,\dots,n\}$ *be* GOS*s based the continuous distribution function* F *of a nonnegative random variable with* $m_i \ge -1$ *for each* i*. If* F *is* DFR*, then*

$$\left(0, X_{(1,n,\tilde{m}_n,k)},\dots,X_{(n-1,n,\tilde{m}_n,k)}\right) \le_{\text{disp}} \left(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)},\dots,X_{(n,n,\tilde{m}_n,k)}\right). \tag{2.4}$$

Proof. Define

$$\mathbf{X} = \left(0, X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n-1,n,\tilde{m}_n,k)}\right)$$

and

$$\mathbf{Y} = \left(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right).$$

Since

$$G_1(y) = \mathbb{P}\left(X_{(1,n,\tilde{m}_n,k)} \le y\right) = 1 - \left[\overline{F}(y)\right]^{\gamma_{1,n}}, \quad \forall\, y \in \Re,$$

it follows that

$$y_1(\mathbf{u}) = G_1^{-1}(u_1) = F^{-1}\left(1 - \bar{u}_1^{1/\gamma_{1,n}}\right), \quad \mathbf{u} \in (0,1)^n.$$

Here and henceforth, we use the notation

$$\bar{u}_i = 1 - u_i \text{ for each } i \text{ and } u_i \in (0,1). \tag{2.5}$$

From the Markovian property of GOSs, we get that, for $r = 2, \ldots, n$,

$$\begin{aligned} y_r(\mathbf{u}) &= G^{-1}_{r|1,\ldots,r-1}(u_r|y_1, \ldots, y_{r-1}) \\ &= F^{-1}\left(1 - \overline{F}(y_{r-1})\, \bar{u}_r^{1/\gamma_{r,n}}\right), \quad \mathbf{u} \in (0,1)^n. \end{aligned}$$

Define

$$H(x) = F^{-1}\left(1 - e^{-x}\right), \quad \forall\, x \in \Re_+. \tag{2.6}$$

Then, for $r = 1, \ldots, n$,

$$y_r(\mathbf{u}) = H\left(-\sum_{i=1}^{r} \frac{1}{\gamma_{i,n}} \log \bar{u}_i\right), \quad \mathbf{u} \in (0,1)^n.$$

Similarly, $x_1(\mathbf{u}) = F_1^{-1}(u_1) = 0$ and

$$\begin{aligned} x_r(\mathbf{u}) &= F^{-1}_{r|1,\ldots,r-1}(u_r|x_1, \ldots, x_{r-1}) \\ &= H\left(-\sum_{i=2}^{r} \frac{1}{\gamma_{i-1,n}} \log \bar{u}_i\right), \quad \mathbf{u} \in (0,1)^n, \end{aligned}$$

for $r = 2, \ldots, n$.

To prove (2.4), we have to prove that for each $r \ge 2$,

$$y_r(\mathbf{u}) - x_r(\mathbf{u}) = H\left(-\sum_{i=1}^{r} \frac{1}{\gamma_{i,n}} \log \bar{u}_i\right) - H\left(-\sum_{i=2}^{r} \frac{1}{\gamma_{i-1,n}} \log \bar{u}_i\right)$$

is increasing in $(u_2, \ldots, u_r) \in (0,1)^{r-1}$. Fix an integer j, $2 \le j \le r$, and $\mathbf{u} \in (0,1)^n$. Observe that $m_i \ge -1$ for each i,

$$\gamma_{i,n} = k + \sum_{\nu=i}^{n-1} (m_\nu + 1) \ge k + \sum_{\nu=i+1}^{n-1} (m_\nu + 1) = \gamma_{i+1,n}, \quad i = 1, \ldots, n-1,$$

and that F being DFR implies that $H(x)$ is strictly increasing and convex. Then, denoting

$$a_{r,j}(\mathbf{u}) = -\sum_{1 \le i \le r, i \ne j} \frac{1}{\gamma_{i,n}} \log \bar{u}_i, \quad b_{r,j}(\mathbf{u}) = -\sum_{2 \le i \le r, i \ne j} \frac{1}{\gamma_{i-1,n}} \log \bar{u}_i,$$

we have that for each $u_j^* \in (u_j, 1)$,

$$\begin{aligned} & a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}} \log \bar{u}_j^* \\ & \quad \ge \left[a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}} \log \bar{u}_j,\ b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}} \log \bar{u}_j^* \right] \\ & \quad \ge b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}} \log \bar{u}_j \end{aligned}$$

and

$$\begin{aligned} & \left[a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}} \log \bar{u}_j^* \right] + \left[b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}} \log \bar{u}_j \right] \\ & \qquad \ge \left[a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}} \log \bar{u}_j \right] + \left[b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}} \log \bar{u}_j^* \right], \end{aligned}$$

where we use the notation: for any $s, t, v \in \Re$, $[s,t] \le v$ means that $s \le v$ and $t \le v$, and $v \ge [s,t]$ means that $v \ge s$ and $v \ge t$. Thus, by using the convexity and monotonicity of H, we have

$$\begin{aligned} & H\left(a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}} \log \bar{u}_j^* \right) + H\left(b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}} \log \bar{u}_j \right) \\ & \quad \ge H\left(a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}} \log \bar{u}_j \right) + H\left(b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}} \log \bar{u}_j^* \right) \end{aligned}$$

and, hence,

$$y_r(u_j^*; \mathbf{u}^{(j)}) - x_r(u_j^*; \mathbf{u}^{(j)})$$

$$\begin{aligned} &= H\left(a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}}\log \bar{u}_j^*\right) - H\left(b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}}\log \bar{u}_j^*\right) \\ &\geq H\left(a_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j,n}}\log \bar{u}_j\right) - H\left(b_{r,j}(\mathbf{u}) - \frac{1}{\gamma_{j-1,n}}\log \bar{u}_j\right) \\ &= y_r(\mathbf{u}) - x_r(\mathbf{u}), \end{aligned}$$

where $y_r(u_j^*; \mathbf{u}^{(j)})$ and $x_r(u_j^*; \mathbf{u}^{(j)})$ are the values of functions $y_r(\cdot)$ and $x_r(\cdot)$ at point $\mathbf{u}$ with jth component being u_j^*. This means that $y_r(\mathbf{u}) - x_r(\mathbf{u})$ is increasing in u_j. Thus, we complete the proof of the theorem. ∎

Under the conditions in Theorem 2.1, it is difficult to prove that

$$\left(X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n-1,n,\tilde{m}_n,k)}\right) \leq_{\text{disp}} \left(X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right). \tag{2.7}$$

Moreover, we do not know whether (2.7) holds.

Theorem 2.2. *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ *and* $\{X_{(r,n+1,\tilde{m}_{n+1},k)}, r = 1, \ldots, n+1\}$ *be* GOS*s based on the distribution function* F *with* $F(0) = 0$, $\tilde{m}_{n+1} = (\tilde{m}_n, m_n)$ *and* $m_i \geq -1$ *for each* i. *If* F *is* DFR*, then*

$$\begin{aligned} &\left(X_{(1,n+1,\tilde{m}_{n+1},k)}, X_{(2,n+1,\tilde{m}_{n+1},k)}, \ldots, X_{(n,n+1,\tilde{m}_{n+1},k)}\right) \\ &\qquad \leq_{\text{disp}} \left(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right). \end{aligned} \tag{2.8}$$

Proof. Denote by $\mathbf{X}$ and $\mathbf{Y}$ the vectors in the left and right hand sides of (2.8). Then

$$y_r(\mathbf{u}) = G^{-1}_{r|1,\ldots,r-1}(u_r|y_1, \ldots, y_{r-1}) = H\left(-\sum_{i=1}^{r}\frac{1}{\gamma_{i,n}}\log \bar{u}_i\right)$$

and

$$x_r(\mathbf{u}) = F^{-1}_{r|1,\ldots,r-1}(u_r|x_1, \ldots, x_{r-1}) = H\left(-\sum_{i=1}^{r}\frac{1}{\gamma_{i,n+1}}\log \bar{u}_i\right)$$

for $\mathbf{u} \in (0,1)^n$ and $r = 1, \ldots, n$. Observing that $\gamma_{i,n+1} \geq \gamma_{i,n}$ for $i = 1, \ldots, n$, the rest of the proof is similar to that of Theorem 2.1 and, hence, omitted. ∎

Theorem 2.3. *Under the same conditions as in Theorem* 2.2. *If, in addition,* $m_n \le \min_{1\le j\le n-1} m_j$, *then*

$$\Big(0, X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\Big) \\ \le_{\text{disp}} \Big(X_{(1,n+1,\tilde{m}_{n+1},k)}, X_{(2,n+1,\tilde{m}_{n+1},k)}, \ldots, X_{(n+1,n+1,\tilde{m}_{n+1},k)}\Big). \quad (2.9)$$

Proof. Denote by $\mathbf{X}$ and $\mathbf{Y}$ the vectors in the left and right hand sides of (2.9). Then

$$\begin{aligned} y_r(\mathbf{u}) &= G^{-1}_{r|1,\ldots,r-1}(u_r|y_1,\ldots,y_{r-1}) \\ &= H\left(-\sum_{i=1}^{r} \frac{1}{\gamma_{i,n+1}} \log \bar{u}_i\right), \quad \mathbf{u} \in (0,1)^{n+1}, \quad r \ge 1, \end{aligned}$$

and $x_1(\mathbf{u}) = 0$ and

$$\begin{aligned} x_r(\mathbf{u}) &= F^{-1}_{r|1,\ldots,r-1}(u_r|x_1,\ldots,x_{r-1}) \\ &= H\left(-\sum_{i=2}^{r} \frac{1}{\gamma_{i-1,n}} \log \bar{u}_i\right), \quad \mathbf{u} \in (0,1)^{n+1}, \quad r \ge 2. \end{aligned}$$

Since $m_n \le \min_{1\le j\le n-1} m_j$, it follows that

$$\gamma_{i-1,n} = k + \sum_{j=i-1}^{n-1}(m_j+1) \ge k + \sum_{j=i}^{n}(m_j+1) = \gamma_{i,n+1}$$

for $i = 2, \ldots, n+1$. Thus, the rest of the proof is similar to that of Theorem 2.1 and, hence, omitted. ∎

Combining Theorems 2.1, 2.2 and 2.3 with Theorem 1.1, we can obtain multivariate dispersive ordering between GOSs from two samples.

Theorem 2.4. *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ *and* $\{Y_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ *be* GOS*s based on two continuous distribution functions* F *and* G*, respectively, with* $m_i \ge -1$ *for each* i *and* $F(0) = G(0) = 0$. *If either* F *or* G *is* DFR, *and* $F \le_{\text{disp}} G$, *then*

$$\Big(0, X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n-1,n,\tilde{m}_n,k)}\Big) \\ \le_{\text{disp}} \Big(Y_{(1,n,\tilde{m}_n,k)}, Y_{(2,n,\tilde{m}_n,k)}, \ldots, Y_{(n,n,\tilde{m}_n,k)}\Big).$$

Theorem 2.5. *Let* $\{X_{(r,n+1,\tilde{m}_{n+1},k)}, r=1,\ldots,n+1\}$ *and* $\{Y_{(r,n,\tilde{m}_n,k)}, r=1,\ldots,n\}$ *be* GOS*s based on two continuous distribution functions* F *and* G*, respectively, with* $\tilde{m}_{n+1}=(\tilde{m}_n, m_n)$*, and* $m_i \geq -1$ *for each* i *and* $F(0)=G(0)=0$*. If either* F *or* G *is* DFR*, and* $F \leq_{\rm disp} G$*, then*

$$\left(X_{(1,n+1,\tilde{m}_{n+1},k)}, X_{(2,n+1,\tilde{m}_{n+1},k)}, \ldots, X_{(n,n+1,\tilde{m}_{n+1},k)}\right) \\ \leq_{\rm disp} \left(Y_{(1,n,\tilde{m}_n,k)}, Y_{(2,n,\tilde{m}_n,k)}, \ldots, Y_{(n,n,\tilde{m}_n,k)}\right).$$

Theorem 2.6. *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r=1,\ldots,n\}$ *and* $\{Y_{(r,n+1,\tilde{m}_{n+1},k)}, r=1,\ldots,n+1\}$ *be* GOS*s based on two continuous distribution functions* F *and* G*, respectively, with* $\tilde{m}_{n+1}=(\tilde{m}_n, m_n)$*, and* $m_i \geq -1$ *for each* i *and* $F(0)=G(0)=0$*. If either* F *or* G *is* DFR *and* $F \leq_{\rm disp} G$*, and if* $m_n \leq \min_{j=1}^{n-1} m_j$*, then*

$$\left(0, X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right) \\ \leq_{\rm disp} \left(Y_{(1,n+1,\tilde{m}_{n+1},k)}, Y_{(2,n+1,\tilde{m}_{n+1},k)}, \ldots, Y_{(n+1,n+1,\tilde{m}_{n+1},k)}\right).$$

Note that, in the univariate case, there is the following intimate connection between the hazard rate order and the dispersive order (see Bartoszewicz, 1985; Bagai and Kochar, 1986). For two distribution functions F and G, F is said to be smaller than G in the hazard rate order, denoted by $F \leq_{\rm hr} G$, if $\overline{G}(x)/\overline{F}(x)$ is increasing in x. If $F \leq_{\rm hr} G$ with $F(0)=G(0)=0$, and if either F or G is DFR, then $F \leq_{\rm disp} G$. Thus, the condition $F \leq_{\rm disp} G$ in Theorems 2.4, 2.5 and 2.6 can be replaced by $F \leq_{\rm hr} G$.

Finally, we present one result concerning comparison of GOSs from two samples in the sense of Khaledi-Kochar multivariate dispersive order. Recall that a copula is a multivariate distribution function with uniform margins on $(0,1)$. Given a multivariate distribution function F with margins as $F_1,\ldots,F_n$, there exists a copula C such that

$$F(\mathbf{x}) = C(F_1(x_1),\ldots,F_n(x_n)) \quad \text{for all } \mathbf{x} \in \Re^n.$$

Moreover, if F is continuous, then C is unique and can be constructed as follows:

$$C(\mathbf{u}) = F(F_1^{-1}(u_1),\ldots,F_n^{-1}(u_n)) \quad \text{for all } \mathbf{u} \in (0,1)^n$$

(see Nelsen, 1999).

Lemma 2.1. [Khaledi & Kochar, 2005] *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* n*-dimensional random vectors with a common copula. Then* $\mathbf{X} \leq_{\text{uo}-\text{disp}} \mathbf{Y}$ *if and only if* $X_i \leq_{\text{disp}} Y_i$ *for each* $i = 1, \ldots, n$.

Theorem 2.7. *Let* $\{X_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ *and* $\{Y_{(r,n,\tilde{m}_n,k)}, r = 1, \ldots, n\}$ *be* GOS*s based on continuous distribution functions* F *and* G*, respectively. If* $F \leq_{\text{disp}} G$*, then*

$$\begin{aligned}&\left(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right)\\ &\qquad \leq_{\text{uo}-\text{disp}} \quad \left(Y_{(1,n,\tilde{m}_n,k)}, Y_{(2,n,\tilde{m}_n,k)}, \ldots, Y_{(n,n,\tilde{m}_n,k)}\right). \quad (2.10)\end{aligned}$$

Proof. By the definition of GOSs, we have that

$$\left(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right)$$

and

$$\left(Y_{(1,n,\tilde{m}_n,k)}, Y_{(2,n,\tilde{m}_n,k)}, \ldots, Y_{(n,n,\tilde{m}_n,k)}\right)$$

have a common copula (see Nelsen, 1999, p.22). On the other hand, from Theorem 3.12 in Belzunce et al. (2005), we get that

$$X_{(i,n,\tilde{m}_n,k)} \leq_{\text{disp}} Y_{(i,n,\tilde{m}_n,k)}, \quad i = 1, \ldots, n.$$

Therefore, (2.10) follows from Lemma 2.1. ∎

3 Applications

Before we present some applications, we need some definitions.

Definition 3.1. *For a function* $\phi : \Re^n \to \Re$*, define the difference operators*

$$\Delta_i^\epsilon \phi(\mathbf{x}) = \phi(\mathbf{x} + \epsilon \mathbf{e}_i) - \phi(\mathbf{x}),$$

where $\mathbf{e}_i$ *is the* i*th unit vector with respect to the canonical base of* $\Re^n$ *and* $\epsilon > 0$*.* ϕ *is said to be directionally convex if*

$$\Delta_i^\epsilon \Delta_j^\delta \phi(\mathbf{x}) \geq 0$$

holds for all $\mathbf{x} \in \Re^n$*,* $1 \leq i, j \leq n$ *and all* $\epsilon, \delta > 0$*.*

Müller and Scarsini (2001) gave several equivalent conditions to characterize directionally convex functions (see also Shaked and Shanthikumar, 1990). For example, ϕ is directionally convex if and only if

$$\phi(\mathbf{x}_2) + \phi(\mathbf{x}_3) \leq \phi(\mathbf{x}_1) + \phi(\mathbf{x}_4)$$

for all $\mathbf{x}_i \in \Re^n$, $i = 1, 2, 3, 4$, such that $\mathbf{x}_1 \leq \mathbf{x}_2 \leq \mathbf{x}_4$, $\mathbf{x}_1 \leq \mathbf{x}_3 \leq \mathbf{x}_4$ and $\mathbf{x}_1 + \mathbf{x}_4 = \mathbf{x}_2 + \mathbf{x}_3$. Directionally convexity neither implies, nor is implied by, conventional convexity. If ϕ is twice differentiable then it is directionally convex if and only if all its second derivatives are nonnegative. For examples of directionally convex, one may refer to Kulik (2003).

Shaked and Shanthikumar (1998) introduce condition $\mathbf{X} \leq_{\rm disp} \mathbf{Y}$ to identify pairs of multivariate functions $\phi(\mathbf{X})$ and $\phi(\mathbf{Y})$ of $\mathbf{X}$ and $\mathbf{Y}$ that are ordered in the st:icx order. A random variable X is said to be smaller than Y in the st:icx order, denoted by $X \leq_{\rm st:icx} Y$, if $\mathbb{E}[h(X)] \leq \mathbb{E}[h(Y)]$ for all increasing functions h, and if

$$\mathrm{Var}\,(h(X)) \leq \mathrm{Var}\,(h(Y))$$

for all increasing convex functions h. Recall that a random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be CIS (*Conditionally Increasing in Sequence*) if $\mathbb{P}(X_i > t | X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ is increasing in $(x_1, \ldots, x_{i-1})$ for each $i = 2, \ldots, n$ and each t.

Proposition 3.1. [Shaked and Shanthikumar, 1998] *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two n-dimensional nonnegative* CIS *random vectors. If* $\mathbf{X} \leq_{\rm disp} \mathbf{Y}$, *then*

$$\phi(\mathbf{X}) \leq_{\rm st:icx} \phi(\mathbf{Y})$$

for all increasing directionally convex functions $\phi : \Re^n_+ \to \Re$.

From (2.3), it follows that GOSs have the CIS property. Combining Theorems 2.1–2.6 with Proposition 3.1, we can get several interesting corollaries. We just list three ones corresponding to Theorems 2.1, 2.2 and 2.3.

Corollary 3.1. *Under the same conditions as in Theorem* 2.1, *we have that*

$$\phi\left(X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n-1,n,\tilde{m}_n,k)}\right) \\ \leq_{\rm st:icx} \phi\left(X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right)$$

for all increasing and directionally convex functions $\phi : \Re^{n-1}_+ \to \Re$.

Corollary 3.2. *Under the same conditions as in Theorem* 2.2*, we have that*

$$\phi\left(X_{(1,n+1,\tilde{m}_{n+1},k)}, X_{(2,n+1,\tilde{m}_{n+1},k)}, \ldots, X_{(n,n+1,\tilde{m}_{n+1},k)}\right) \\ \leq_{\text{st:icx}} \phi\left(X_{(1,n,\tilde{m}_n,k)}, X_{(2,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right)$$

for all increasing and directionally convex functions $\phi : \Re_+^n \to \Re$.

Corollary 3.3. *Under the same conditions as in Theorem* 2.3*, we have that*

$$\phi\left(X_{(1,n,\tilde{m}_n,k)}, \ldots, X_{(n,n,\tilde{m}_n,k)}\right) \\ \leq_{\text{st:icx}} \phi\left(X_{(2,n+1,\tilde{m}_{n+1},k)}, \ldots, X_{(n+1,n+1,\tilde{m}_{n+1},k)}\right)$$

for all increasing and directionally convex functions $\phi : \Re_+^n \to \Re$.

The next proposition states an interesting property of the Khaledi-Kochar multivariate dispersive order.

Proposition 3.2. [Khaledi and Kochar, 2005] *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two* n*-dimensional random vectors such that*

$$\left[X_i \,\middle|\, \bigcap_{j\neq i}\{X_j > F_j^{-1}(u_j)\}\right] \quad \text{and} \quad \left[Y_i \,\middle|\, \bigcap_{j\neq i}\{Y_j > G_j^{-1}(u_j)\}\right]$$

have a common finite left endpoint of their supports for all $\mathbf{u} \in (0,1)^n$ *and each* i*. If* $\mathbf{X} \leq_{\text{uo}-\text{disp}} \mathbf{Y}$*, and*

$$u_i u_j \leq C_{ij}^{\mathbf{X}}(u_i, u_j) \leq C_{ij}^{\mathbf{Y}}(u_i, u_j), \quad \forall\ (u_i, u_j) \in (0,1)^2,\ i \neq j,$$

then

$$\text{Cov}\left(\phi_1(X_i), \phi_2(X_j)\right) \leq \text{Cov}\left(\phi_1(Y_i), \phi_2(Y_j)\right)$$

holds for all increasing convex functions ϕ_1 *and* ϕ_2*, where* $C_{ij}^{\mathbf{X}}(\cdot)$ *and* $C_{ij}^{\mathbf{Y}}(\cdot)$ *are the copulas corresponding to* (X_i, X_j) *and* (Y_i, Y_j).

An immediate consequence of Theorem 2.7 and Proposition 3.2 is

Corollary 3.4. *Under the same conditions as in Theorem* 2.7*, we have that*

$$\text{Cov}\left(\phi_1\left(X_{(i,n,\tilde{m}_n,k)}\right), \phi_2\left(X_{(j,n,\tilde{m}_n,k)}\right)\right) \\ \leq \text{Cov}\left(\phi_1\left(Y_{(i,n,\tilde{m}_n,k)}\right), \phi_2\left(Y_{(j,n,\tilde{m}_n,k)}\right)\right)$$

holds for all $1 \leq i \neq j \leq n$ *and increasing convex functions* ϕ_1 *and* ϕ_2.

Since GOSs contain several models of random vectors with ordered components, we can apply the previous results to these models. For example, choosing $k = 1$ and $m_i = -1$ for each i, it follows from Corollary 3.3 that

Corollary 3.5. *Let $X_{L(1)}, X_{L(2)}, \ldots$ be upper record values based on a sequence of independent and identically distributed nonnegative random variables with continuous distribution function F. If F is* DFR*, then*

$$\phi\left(X_{L(1)}, \ldots, X_{L(n)}\right) \leq_{\text{st:icx}} \phi\left(X_{L(2)}, \ldots, X_{L(n+1)}\right)$$

for all increasing and directionally convex functions $\phi : \Re^n_+ \to \Re$.

In Corollary 3.5, $\{X_{L(1)}, X_{L(2)}, \ldots\}$ can be interpreted as the epoch times $\{T_1, T_2, \ldots\}$ of a nonhomogeneous Poisson process with intensity function $\lambda(t)$, where $\lambda(t)$ is the failure rate of F.

References

Arias-Nicolás, J. P., Fernádez-Ponse, J. M., Luque-Calvo, P., and Suárez-Llorens, A. (2005). Multivariate dispersion order and the notion of copula applied to the multivariate t-distribution. Probability in the Engineering and Informational Sciences, **19**, 363-375.

Bagai, I. and Kochar, S. C. (1986), On tail-ordering and comparison of failure rates. Communications in Statistics –Theory and Methods, **15**, 1377-1388.

Bartoszewicz, J. (1985), Dispersive ordering and monotone failure rate distributions. Advances in Applied Probability, **17**, 472-474.

Belzunce, F., Mercader, J. A., and Ruiz, J. M. (2005), Stochastic comparisons of generalized order statistics. Probability in the Engineering and Informational Sciences, **19**, 99-120.

Belzunce, F. and Ruiz, J. M. (2002), Multivariate dispersive ordering of epoch times of nonhomogeneous Poisson processes. Journal of Applied Probability, **39**, 637-643.

Belzunce, F., Ruiz, J. M., and Ruiz, M. C. (2003), Multivariate properties of random vectors of order statistics. Journal of Statistical Planning and Inference, **115**, 413-424.

Fernádez-Ponse, J. M. and Suárez-Lloréns, A. (2003), A multivariate dispersion ordering based on quantiles more widely separated. Journal of Multivariate Analysis, **85**, 40-53.

Fernádez-Ponse, J. M. and Rodríguez-Griñolo (2006), Preserving multivariate dispersion: An application to the Wishart distribution. Journal of Multivariate Analysis, **97**, 1208-1220.

Giovagnoli, A. and Wynn, H. P. (1995), Multivariate dispersive orderings. Statistics & Probability Letters, **22**, 325-332.

Khaledi, B.-E. and Kochar, S. (2005), Dependence, dispersiveness and multivariate hazard rate ordering. Probability in the Engineering and Informational Sciences, **19**, 427-446.

Kamps, U. (1995a), A Concept of Generalized Order Statistics. Stuttgard: Teubner.

Kulik, R. (2003), Stochastic comparison of multivariate random sums. Applicationes Mathematicae, **30**, 379-387.

Müller, A. and Scarsini, M. (2001), Stochastic comparison of random vectors with a common copula. Mathematics of Operations Research, **26**, 723-740.

Nelsen, R. B. (1999), An Introduction to Copulas. New York: Springer-Verlag.

Shaked, M. and Shanthikumar, J. G. (1990), Parametric stochastic convexity and concavity of stochastic processes. Annals of the Institute of Statistical Mathematics, **42**, 509-531.

Shaked, M. and Shanthikumar, J. G. (1994), Stochastic Orders and Their Applications. New York: Academic Press.

Shaked, M. and Shanthikumar, J. G. (1998), Two variability orders. Probability in the Engineering and Informational Sciences, **12**, 1-23.