

Matrix Kummer-Pearson VII Relation and Polynomial Pearson VII Configuration Density

Francisco J. Caro-Lopera¹, José A. Díaz-García²

¹Departamento de Ciencias Básicas, Universidad de Medellín, Colombia.

²Departamento de Estadística y Cálculo, Universidad Autónoma Agraria Antonio Narro, México.

Abstract. A case of the matrix Kummer relation of Herz (1955) based on the Pearson VII type matrix model is derived in this paper. As a consequence, the polynomial Pearson VII configuration density is obtained and this sets the corresponding exact inference as a solvable aspect in shape theory. An application in postcode recognition, including a numerical comparison between the exact polynomial and the truncated configuration density, is given at the end of the paper.

Keywords. Matrix Kummer relation, Pearson VII configuration density, zonal polynomials.

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1 Introduction

Start assuming that a given sample of n “figures”, comprised in N landmarks (“anatomical” points) in K dimensions, and summarized in an $N \times K$ matrix \mathbf{X} , belongs to certain matrix variate distribution with unknown scale and location parameters. The statistical theory of shape pursues the distribution of the transforming \mathbf{X} after filtering out some non important geometrical aspect of the original figures, such as the

Francisco J. Caro-Lopera(fjcaro@udem.edu.co), José A. Díaz-García(✉) (jadiaz@uaaan.mx)

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scale, the position, the rotation, the uniform shear, and so on; then the so called shape of the population object should be inferred and some comparisons among population shapes could be performed, among many others statistical comparisons in the shape space, instead of comparisons in the very noised non transformed Euclidean space. As usual, the classical theory of shape, with different geometrical filters, as the Euclidean or affine, was based on Gaussian samples, see for example Goodall and Mardia (1993) and the references therein. Then, the non normal applications demanded general assumptions for the samples, and the so termed generalized shape theory was set under elliptical models. Under the Euclidean filter we can mention the work of Díaz-García and Caro-Lopera (2010) and Díaz-García and Caro-Lopera (2012); they find the density of all geometrical information about the elliptical random \mathbf{X} which remains after removing the scale, the position and the rotation. Finally, if an affine filter is applied, in order to remove from \mathbf{X} all geometrical information of scale, position, rotation and uniform shear, Caro-Lopera *et al*(2010) obtained the so term configuration density of \mathbf{X} . All the above densities are expanded in terms of the well known zonal polynomials, in a series of papers by A.T. James in 60's, see for example Muirhead(1982).

The transition of the Gaussian shape theory to the elliptical shape theory demanded some advances in integration involving zonal polynomials (see for example Caro-Lopera *et al*(2010)), but important problems remain, the computability of the shape densities.

In this paper we focus in alternatives for such problems. It is easy to see in Goodall and Mardia (1993), Díaz-García and Caro-Lopera (2010) and Díaz-García and Caro-Lopera (2012) that the structure of shape densities under Euclidean transformations involves series of zonal polynomials which heritages the difficulties for computations of the classical hypergeometric series studied by Koev and Edelman (2006). However, a class of the generalized confluent type series of the configuration densities of Caro-Lopera *et al*(2010) can be handled in order to transformed the series into polynomials, and then the addressed open problems for computations of the series can be avoided.

The configuration density under Kotz type samples (including Gaussian) has this property, and the inference can be performed with polynomials instead of infinite series, see Caro-Lopera *et al*(2009). The source for this property resides in a generalization of the Kummer relation of Herz (1955).

This motivates the present work, claiming that there is a similar

Kummer relation based on a Pearson VII distribution, which under certain restriction of the parameters in the associated Pearson VII configuration density, it can be turned into a polynomial density. Then the inference can be performed easily by working with the exact likelihood, which is written in terms of very low zonal polynomials. In fact the exact densities can be write down by using formulae for those polynomials; for example, in the planar shape theory (the most classical applications resides in the study of figures in \mathbb{R}^2), we can use directly the formulae given by Caro-Lopera *et al*(2007) instead of the numerical approaches by Koev and Edelman (2006), in order to perform some analytical properties of the exact density.

This discussion is placed in the paper as follows: section 2 defines a Pearson VII type series and finds an integral representation that leads to a matrix Kummer type relation which we call Kummer-Pearson VII relation; then by applying some general properties studied by Herz (1955), the equality is extended for the required domains in the shape theory context. Finally, Kummer-Pearson VII relation gives the finiteness of the Pearson VII configuration density in section 3. Finally, section 4 studies an experiment of handwritten digit 3, and compares numerically the exact and the truncated associated estimates of the mean configurations.

2 Matrix Kummer-Pearson VII relation

Recall that the matrix Kummer relation (due to Herz (1955), see also Muirhead(1982)) states that

$${}_1F_1(a; c; \mathbf{X}) = \text{etr}(\mathbf{X}) {}_1F_1(c - a; c; -\mathbf{X}). \tag{1}$$

Now, let $\mathbf{X} > 0$ be an $m \times m$ positive definite matrix, then define

$${}_1P_1(f(t, \mathbf{X}) : a; c; \mathbf{X}) = \sum_{t=0}^{\infty} \frac{f(t, \mathbf{X})}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} C_{\tau}(\mathbf{X}), \tag{2}$$

where the function $f(t, \mathbf{X})$ is independent of τ , $\tau = (t_1, \dots, t_m)$, $t_1 \geq t_2 \dots \geq t_m > 0$, is a partition of t ,

$$(\beta)_{\tau} = \prod_{i=1}^m \left(\beta - \frac{1}{2}(i - 1) \right)_{t_i},$$

and

$$(b)_t = b(b + 1) \dots (b + t - 1), \quad (b)_0 = 1.$$

Then, using this notation we see that the Kummer relation (1) is a particular case of a general type of expressions with the following form

$${}_1P_1(f(t, \mathbf{X}) : a; c; \mathbf{X}) = v(\mathbf{X}) {}_1P_1(g(t, \mathbf{X}) : c - a; c; h(\mathbf{X})), \quad (3)$$

where the functions v, g and h are uniquely determined by the particular function f and according to the domain of the parameters a, c and the matrix \mathbf{X} .

First, we consider an integral representation of the left hand side of (3) under the model $f(t, \mathbf{X}) = (b)_t$.

Theorem 2.1. *Let $\mathbf{X} < \mathbf{I}$, $Re(a) > (m - 1)/2$, $Re(c) > (m - 1)/2$ and $Re(c - a) > (m - 1)/2$. Then for suitable reals b and d , we have that*

$$\begin{aligned}
 {}_1P_1\left((b)_t d^{-b-t} : a; c; \mathbf{X}\right) &= \frac{\Gamma_m[c]}{\Gamma_m[a]\Gamma_m[c-a]} \\
 &\times \int_{0 < \mathbf{Y} < \mathbf{I}_m} (d - \text{tr}(\mathbf{X}\mathbf{Y}))^{-b} |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I} - \mathbf{Y}|^{c-a-(m+1)/2} (d\mathbf{Y}). \quad (4)
 \end{aligned}$$

Proof. First, we use a zonal polynomial expansion

$$\begin{aligned}
 (d - \text{tr}(\mathbf{X}\mathbf{Y}))^{-b} &= \sum_{t=0}^{\infty} \frac{(b)_t d^{-b-t}}{t!} [\text{tr}(\mathbf{X}\mathbf{Y})]^t \\
 &= \sum_{t=0}^{\infty} \frac{(b)_t d^{-b-t}}{t!} \sum_{\tau} C_{\tau}(\mathbf{X}\mathbf{Y}).
 \end{aligned}$$

Then integrating term by term using [Muirhead, 1982, theorem 7.2.10], we have that

$$\begin{aligned}
 &\int_{0 < \mathbf{Y} < \mathbf{I}_m} (d - \text{tr}(\mathbf{X}\mathbf{Y}))^{-b} |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I} - \mathbf{Y}|^{c-a-(m+1)/2} (d\mathbf{Y}) \\
 &= \sum_{t=0}^{\infty} \frac{(b)_t d^{-b-t}}{t!} \sum_{\tau} \int_{0 < \mathbf{Y} < \mathbf{I}_m} |\mathbf{Y}|^{a-(m+1)/2} \\
 &\quad \times |\mathbf{I} - \mathbf{Y}|^{c-a-(m+1)/2} C_{\tau}(\mathbf{X}\mathbf{Y}) (d\mathbf{Y}) \\
 &= \sum_{t=0}^{\infty} \frac{(b)_t d^{-b-t}}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} \frac{\Gamma_m[a]\Gamma_m[c-a]}{\Gamma_m[c]} C_{\tau}(\mathbf{X}) \\
 &= \frac{\Gamma_m[a]\Gamma_m[c-a]}{\Gamma_m[c]} \sum_{t=0}^{\infty} \frac{(b)_t d^{-b-t}}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} C_{\tau}(\mathbf{X}) \\
 &= \frac{\Gamma_m[a]\Gamma_m[c-a]}{\Gamma_m[c]} {}_1P_1\left((b)_t d^{-b-t} : a; c; \mathbf{X}\right),
 \end{aligned}$$

and the required result follows. □

Now, we derive the version of (1) but based on a Pearson VII type model, we call this expression, Kummer-Pearson VII relation.

Theorem 2.2. *Let $\mathbf{X} > 0$, $Re(a) > (m-1)/2$, $Re(c) > (m-1)/2$ and $Re(c-a) > (m-1)/2$. Then for suitable reals b and d , the Kummer-Pearson VII relation is given by*

$$\begin{aligned}
 {}_1P_1 \left((b)_t d^{-b-t} : a; c; \mathbf{X} \right) \\
 = (d - \text{tr } \mathbf{X})^{-b} {}_1P_1 \left((b)_t (d - \text{tr } \mathbf{X})^{-t} : c - a; c; -\mathbf{X} \right). \tag{5}
 \end{aligned}$$

Proof. Consider $\mathbf{W} = \mathbf{I} - \mathbf{Y}$ in (4), then we obtain

$$\begin{aligned}
 {}_1P_1 \left((b)_t d^{-b-t} : a; c; \mathbf{X} \right) &= \frac{\Gamma_m[c]}{\Gamma_m[a]\Gamma_m[c-a]} \\
 &\times \int_{0 < \mathbf{W} < \mathbf{I}_m} (d - \text{tr}[\mathbf{X}(\mathbf{I} - \mathbf{W})])^{-b} |\mathbf{W}|^{c-a-(m+1)/2} \\
 &\times |\mathbf{I} - \mathbf{W}|^{a-(m+1)/2} (d\mathbf{W}) \\
 &= \frac{\Gamma_m[c]}{\Gamma_m[a]\Gamma_m[c-a]} \\
 &\times \int_{0 < \mathbf{W} < \mathbf{I}_m} (d - \text{tr } \mathbf{X} - \text{tr}(-\mathbf{X}\mathbf{W}))^{-b} |\mathbf{W}|^{c-a-(m+1)/2} \\
 &\times |\mathbf{I} - \mathbf{W}|^{a-(m+1)/2} (d\mathbf{W}) \\
 &= \frac{\Gamma_m[c]}{\Gamma_m[a]\Gamma_m[c-a]} \\
 &\times \frac{\Gamma_m[a]\Gamma_m[c-a]}{\Gamma_m[c]} {}_1P_1 \left((b)_t (d - \text{tr } \mathbf{X})^{-b-t} : c - a; c; -\mathbf{X} \right),
 \end{aligned}$$

which is the required result. □

The reader can compare theorem 2.2 (and its proof) with the Kummer relation (and its proof given by Herz (1955)). So the analysis of Herz (1955) for extending the above relations for other values of the parameters, holds in the Kummer-Pearson VII relation too.

Explicitly, we proved that the integral representation (4) of ${}_1P_1 \left((b)_t : a; c; \mathbf{X} \right)$ holds for $Re(\mathbf{X}) < \mathbf{I}$ (by analytic continuation), $Re(a) > (m-1)/2$, $Re(c) > (m-1)/2$ and $Re(c-a) > (m-1)/2$. Then by a suitable modification of the arguments in Herz (1955), we can extend the domain of (5) as follows.

Theorem 2.3. $Re(\mathbf{X}) > 0, Re(a) > (m-1)/2$ and $Re(c) > (m-1)/2$. Then for suitable complex numbers b and d , the Kummer-Pearson VII relation is given by

$$\begin{aligned}
 {}_1P_1\left((b)_t d^{-b-t} : a; c; \mathbf{X}\right) \\
 = (d - \text{tr } \mathbf{X})^{-b} {}_1P_1\left((b)_t (d - \text{tr } \mathbf{X})^{-t} : c - a; c; -\mathbf{X}\right). \quad (6)
 \end{aligned}$$

The above relation is important in shape theory applications, in the so called *polynomial Pearson VII configuration density*.

3 Polynomial Pearson VII Configuration Density

Our motivation for studying finite shape densities, comes from the computations of hypergeometric series type involved in these distributions. It is known, that the zonal polynomials are computable very fast by Koev and Edelman (2006), but the problem now resides in the convergence and the truncation of the series of zonal polynomials. In fact, in the same reference of Koev and Edelman (2006) we read:

“Several problems remain open, among them automatic detection of convergence and it is unclear how to tell when convergence sets in. Another open problem is to determine the best way to truncate the series.”

Thus the implicit numerical difficulties for truncation of any configuration density motivate two areas of investigation: first, continue the numerical approach started by (Koev and Edelman (2006)) with the confluent hypergeometric functions and extend it to the case of some configuration series type, as Pearson VII, Bessel, Logistic, for example; or second, propose a theoretical approach for solving the problem analytically (see Caro-Lopera *et al*(2009)).

We study now the second question corresponding to the polynomial Pearson VII configuration density.

Recall that a $p \times n$ random matrix \mathbf{X} is said to have a matrix variate symmetric Pearson type VII distribution with parameters $s, R \in \mathfrak{R}$, $M : p \times n$, $\Sigma : p \times p$, $\Phi : n \times n$ with $R > 0$, $s > np/2$, $\Sigma > 0$, and $\Phi > 0$ if its probability density function is

$$\frac{\Gamma[s]}{(\pi R)^{np/2} \Gamma[s - np/2] |\Sigma|^{n/2} |\Phi|^{p/2}} \left(1 + \frac{\text{tr}(\mathbf{X} - \mathbf{M})' \Sigma^{-1} (\mathbf{X} - \mathbf{M}) \Phi^{-1}}{R} \right)^{-s}.$$

When $s = (np + R)/2$, \mathbf{X} is said to have a matrix variate t -distribution with R degrees of freedom. And in this case, if $R = 1$, then \mathbf{X} is said to have a matrix variate Cauchy distribution, see Caro-Lopera *et al*(2010).

Then, by Caro-Lopera *et al*(2010) we have that (see Goodall and Mardia (1993) for the gaussian case),

Lemma 3.1. *Let be*

$$A = \frac{\Gamma_K [(N - 1)/2]}{\pi^{Kq/2} |\Sigma|^{K/2} |\mathbf{U}'\Sigma^{-1}\mathbf{U}|^{(N-1)/2} \Gamma_K [K/2]}, \quad a = (N - 1)/2, \quad (7)$$

$$\mathbf{X} = \mathbf{U}'\Sigma^{-1}\mu\mu'\Sigma^{-1}\mathbf{U}(\mathbf{U}'\Sigma^{-1}\mathbf{U})^{-1}/R, \quad c = K/2, \quad (8)$$

$$b = s - K(N - 1)/2, \quad d = 1 + \text{tr}(\mu'\Sigma^{-1}\mu) / R. \quad (9)$$

If $\mathbf{Y} \sim E_{N-1 \times K}(\mu_{N-1 \times K}, \Sigma_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, for $\Sigma > 0$, then the non-isotropic noncentral Pearson type VII configuration density is given by

$$A_1 P_1 \left((b)_t d^{-b-t} : a; c; \mathbf{X} \right), \quad (10)$$

where ${}_1P_1(\cdot)$ has been defined in (2).

Unfortunately, the above configuration density with general form $A_1 P_1(f(t) : a; c; \mathbf{X})$ is an infinite series, given that $a = (N - 1)/2$ and $c = K/2$ are positive (recall that N is the number of landmarks, K is de dimension and $N - K - 1 \geq 1$). So a truncation is needed for performing inference when the modified algorithms of Koev and Edelman (2006) are used.

However the above series can be turned into a polynomial if we use the following basic principle of Caro-Lopera *et al*(2009).

Lemma 3.2. *Let be $N - K - 1 \geq 1$ as usual, and consider the definition of ${}_1P_1(\cdot)$ in (2). The infinite configuration density has the general form*

$$CD_1 = w(K, N, \mathbf{X}) {}_1P_1 \left(f(t, \mathbf{X}) : \frac{N - 1}{2}; \frac{K}{2}; \mathbf{X} \right),$$

for suitable functions: $w(\cdot)$, independent of t and τ but dependent of K, N and \mathbf{X} ; and $f(\cdot)$, independent of τ , but dependent of t and possibly of \mathbf{X} (it depends on the generator elliptical function, compare with the particular Pearson VII case of lemma 3.1). Then, according to (3), if the dimension K is even (odd) and the number of landmarks N is odd (even), respectively, then the equivalent configuration density

$$CD_2 = w(K, N, \mathbf{X})v(\mathbf{X}) {}_1P_1 \left(g(t, \mathbf{X}) : - \left(\frac{N - 1}{2} - \frac{K}{2} \right); \frac{K}{2}; h(\mathbf{X}) \right),$$

is a polynomial of degree $K((N - 1)/2 - K/2)$ in the latent roots of the matrix \mathbf{X} (otherwise the series is infinite); where v, g and h are functions understood in the context of (3) and depends on the elliptical generator function.

Given an elliptical configuration density CD_1 indexed by the function $f(\cdot)$ and based in the fact that $a = (N - 1)/2 > 0$, $c = K/2 > 0$, the crucial point here consists of finding an integral representation valid for $c - a = -(N - K - 1)/2 < 0$, which will lead to an equivalent elliptical configuration density CD_2 indexed by some function $g(\cdot)$. Then the finiteness of CD_2 follows from K even (odd) and N odd (even), respectively.

In particular, for the Pearson VII generator function, the referred polynomial density is provided by applying the new Kummer-Pearson VII relation of theorem 2.3, via lemma 3.2, in lemma 3.1.

Theorem 3.1. *Let be A, a, b, c, d and \mathbf{X} defined by (7)-(9).*

If $\mathbf{Y} \sim E_{N-1 \times K}(\mu_{N-1 \times K}, \Sigma_{N-1 \times N-1} \otimes \mathbf{I}_K, h)$, $\Sigma > 0$, K is even (odd) and N is odd (even), respectively, then the polynomial non-isotropic noncentral Pearson type VII configuration density is given by

$$A(d - \text{tr } \mathbf{X})^{-b} {}_1P_1((b)_t (d - \text{tr } \mathbf{X})^{-t} : c - a; c; -\mathbf{X}), \tag{11}$$

and it is a polynomial of degree $K((N - 1)/2 - K/2)$ in the latent roots of \mathbf{X} .

Proof. The proof is trivial, just start with the infinite configuration density (10):

$$A {}_1P_1\left((b)_t d^{-b-t} : a; c; \mathbf{X}\right),$$

where A, a, b, c, d and \mathbf{X} are given by (7)-(9). Then apply (6) and the result follows. Note that the finiteness follows from Lemma (3.2) by noting that $c - a = -(N - K - 1)/2$ is a negative integer, when K is even (odd) and N is odd (even). □

The principle of lemma 3.2 is based on a known property of the hypergeometric series easily extended to series of the type ${}_1P_1(f(t, \mathbf{X}) : a, c; \mathbf{X})$, see (2), which states that if a is a negative integer or a negative half integer, the series vanishes in a polynomial. Then if we have an application following a confluent distribution type, it is a polynomial, always that the parameter a accepts the addressed special domain, otherwise the distribution is a series of zonal polynomials and the open

problems for its computability remains. This last case occurs for example in the general configuration density, which is an infinite confluent series type because $a = (N - 1)/2 > 0$, then it is not trivial to turn that series into polynomials, because the application does not allow a negative parameter. Then the lemma 3.2 gives a special subclass by selecting the number of landmarks (even, odd) given the dimension (odd, even), otherwise the configuration density remains a series, then the associated Kummer relation must be obtained in order to transform the numerator parameter a into $c - a$ which is a negative half integer as required. So, in the context of shape theory under affine transformations, the study of Kummer relation type plays an important role. It is easy to check that the classical Kummer relation first derive by Herz (1955), and set in the context of zonal polynomials by Constantine (1963) is related with a Gaussian kernel; but if we want to obtain some non Gaussian polynomial densities (under the explained restrictions of N and K), then we need to derive new Kummer relations; it was the case of certain class of Kotz configuration densities (which includes the classical Gaussian), it required the derivation of the associated Kummer Kotz relation, see Caro-Lopera *et al*(2009). Thus, in the case of the Pearson polynomial configuration density the corresponding Kummer relation was the key point for transforming series.

The above discussion opens related problems in some special topics of matrix variate analysis involving confluent matrix with special domains for the parameters, or general hypergeometric series type with more or equal Pochhammer symbols in the numerator, several examples of this situations, which demands new developments for Euler relations and similar ones, can be inferred from some distributions proposed in [?, chapters 8–11].

4 An Application in Postcode Recognition

As an illustration of the impressive numerical advantage of the exact polynomial densities against the usual shape densities based on infinite series of zonal polynomials, this section studies some aspects of a classical example in statistical shape theory, the handwritten digit 3. Dryden and Mardia(1998) and other authors have studied the population mean shape of a collection of 30 handwritten number three. The codes were summarized in 13 landmarks, according the curvature of the extremes of the digit. Figure 1 (which was built with the R-package Shapes, authored by I. L. Dryden) shows the sample. The referred studies are based

on the assumption that the landmarks are randomly normal distributed and the shape of the object is reached under filtering out the scale, location and rotation of the figure. However, the nature of the handwritten process seems to be far of a modeling with a similarity transformation and perhaps an affine transformation is closer to this situation, because the main uniform change of the construction of the figure appears predominantly in one direction (vertical), simulating a spring type movement, instead of constant growing in all directions. In that case we remove all geometrical information of the figure which is related with the scale, location, rotation and uniform shear. An statistical proof of this statement can be obtained by using dimension model theory, based on modifications of the so called BIC^* criterion, see Raftery(1995), Rissanen(1978), Yang and Yang(2007), and the references there in. The corresponding statistical proof for this experiment under a Gaussian generator and affine transformation against an Euclidean law is given by Caro-Lopera *et al*(2009).

Table 1: The maximum likelihood estimates of configuration location

	Truncation	Convergence	Time Seg.	Loglikelihood	Iterations	FuncCount
Series	0	N	195	2.2093e+003	15974	20000
Polynomial	0	Y	120.8906	2.1219e+003	8367	10479
Series	5	N	284.5781	2.1686e+003	15977	20000
Polynomial	5	Y	147.4219	1.5346e+003	7854	9914
Series	10	N	361.1563	1.8315e+003	15927	20000
Polynomial	10	Y	228.7500	1.3223e+003	10111	12633
Series	20	Y	378.4844	1.6994e+003	12853	16115
Polynomial	20	Y	299.3594	1.3223e+003	10111	12633
Series	50	Y	532.2656	1.5287e+003	9622	12037
Polynomial	50	Y	521.9375	1.3223e+003	10111	12633
Series	100	Y	1.1054e+003	1.4244e+003	10971	13731
Polynomial	100	Y	883.4219	1.3223e+003	10111	12633
Series	120	Y	957.0781	1.4023e+003	7768	9704
Polynomial	120	Y	901.5687	1.3223e+003	10111	12633
Series	150	Y	1.5775e+003	1.3787e+003	10068	126515
Polynomial	150	Y	1.2637e+003	1.3223e+003	10111	12633
Template						

Tables 1, 2 and 3 shows the configuration location estimates by using two methods, the classical one (based on an infinite series of zonal polynomials, see lemma 3.1) and the polynomial density obtained by the matrix Kummer-Pearson relation (see theorem 3.1); the truncation of the series, the time in seconds and the convergence of the algorithms are given in the corresponding columns; also it is shown the number of iterations and functions being evaluated in the optimization routine and the maximum value reached in the likelihood. Finally, the tables

Table 2: ...Continuation Table 1

	V ₁	V ₂	V ₃	V ₄	V ₅
Series	$\begin{bmatrix} -0.9965 \\ 0.3517 \end{bmatrix}$	$\begin{bmatrix} 0.0709 \\ -0.2434 \end{bmatrix}$	$\begin{bmatrix} 0.0695 \\ -0.3486 \end{bmatrix}$	$\begin{bmatrix} 0.4455 \\ -0.5559 \end{bmatrix}$	$\begin{bmatrix} 0.2338 \\ 0.4746 \end{bmatrix}$
Polynomial	$\begin{bmatrix} 0.0558 \\ 0.6188 \end{bmatrix}$	$\begin{bmatrix} -0.2476 \\ 0.3414 \end{bmatrix}$	$\begin{bmatrix} -0.455 \\ -0.0065 \end{bmatrix}$	$\begin{bmatrix} -0.5717 \\ -0.3417 \end{bmatrix}$	$\begin{bmatrix} -0.435 \\ 0.0312 \end{bmatrix}$
Series	$\begin{bmatrix} -0.4139 \\ 2.1333 \end{bmatrix}$	$\begin{bmatrix} 0.2583 \\ 1.1821 \end{bmatrix}$	$\begin{bmatrix} -1.2376 \\ 1.9376 \end{bmatrix}$	$\begin{bmatrix} -0.7693 \\ -0.3752 \end{bmatrix}$	$\begin{bmatrix} -1.0877 \\ 0.0451 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -1.4363 \\ 1.7525 \end{bmatrix}$	$\begin{bmatrix} -3.0789 \\ 1.5125 \end{bmatrix}$	$\begin{bmatrix} -3.7258 \\ 0.884 \end{bmatrix}$	$\begin{bmatrix} -3.7155 \\ 0.109 \end{bmatrix}$	$\begin{bmatrix} -3.521 \\ 0.763 \end{bmatrix}$
Series	$\begin{bmatrix} 0.0556 \\ 1.0517 \end{bmatrix}$	$\begin{bmatrix} -0.3136 \\ 0.7415 \end{bmatrix}$	$\begin{bmatrix} -0.5988 \\ 0.2595 \end{bmatrix}$	$\begin{bmatrix} -0.7255 \\ -0.2247 \end{bmatrix}$	$\begin{bmatrix} -0.5666 \\ 0.2315 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -1.2968 \\ 2.6833 \end{bmatrix}$	$\begin{bmatrix} -3.1351 \\ 2.2759 \end{bmatrix}$	$\begin{bmatrix} -3.9553 \\ 1.291 \end{bmatrix}$	$\begin{bmatrix} -4.0714 \\ 0.0888 \end{bmatrix}$	$\begin{bmatrix} -3.7561 \\ 1.1051 \end{bmatrix}$
Series	$\begin{bmatrix} 0.0668 \\ 1.3995 \end{bmatrix}$	$\begin{bmatrix} -0.4858 \\ 1.0199 \end{bmatrix}$	$\begin{bmatrix} -0.8513 \\ 0.3873 \end{bmatrix}$	$\begin{bmatrix} -1.049 \\ -0.2859 \end{bmatrix}$	$\begin{bmatrix} -0.8157 \\ 0.324 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -1.2968 \\ 2.6833 \end{bmatrix}$	$\begin{bmatrix} -3.1351 \\ 2.2759 \end{bmatrix}$	$\begin{bmatrix} -3.9553 \\ 1.291 \end{bmatrix}$	$\begin{bmatrix} -4.0714 \\ 0.0888 \end{bmatrix}$	$\begin{bmatrix} -3.7561 \\ 1.1051 \end{bmatrix}$
Series	$\begin{bmatrix} -0.0738 \\ 1.8964 \end{bmatrix}$	$\begin{bmatrix} -0.9106 \\ 1.4216 \end{bmatrix}$	$\begin{bmatrix} -1.4161 \\ 0.5915 \end{bmatrix}$	$\begin{bmatrix} -1.6445 \\ -0.3127 \end{bmatrix}$	$\begin{bmatrix} -1.345 \\ 0.4978 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -1.2968 \\ 2.6833 \end{bmatrix}$	$\begin{bmatrix} -3.1351 \\ 2.2759 \end{bmatrix}$	$\begin{bmatrix} -3.9553 \\ 1.291 \end{bmatrix}$	$\begin{bmatrix} -4.0714 \\ 0.0888 \end{bmatrix}$	$\begin{bmatrix} -3.7561 \\ 1.1051 \end{bmatrix}$
Series	$\begin{bmatrix} -0.3405 \\ 2.216 \end{bmatrix}$	$\begin{bmatrix} -1.456 \\ 1.7225 \end{bmatrix}$	$\begin{bmatrix} -2.0655 \\ 0.7939 \end{bmatrix}$	$\begin{bmatrix} -2.2739 \\ -0.2482 \end{bmatrix}$	$\begin{bmatrix} -1.9498 \\ 0.6722 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -1.2968 \\ 2.6833 \end{bmatrix}$	$\begin{bmatrix} -3.1351 \\ 2.2759 \end{bmatrix}$	$\begin{bmatrix} -3.9553 \\ 1.291 \end{bmatrix}$	$\begin{bmatrix} -4.0714 \\ 0.0888 \end{bmatrix}$	$\begin{bmatrix} -3.7561 \\ 1.1051 \end{bmatrix}$
Series	$\begin{bmatrix} -0.4354 \\ 2.2876 \end{bmatrix}$	$\begin{bmatrix} -1.6332 \\ 1.7975 \end{bmatrix}$	$\begin{bmatrix} -2.2695 \\ 0.8526 \end{bmatrix}$	$\begin{bmatrix} -2.4668 \\ -0.2183 \end{bmatrix}$	$\begin{bmatrix} -2.1407 \\ 0.7234 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -1.2968 \\ 2.6833 \end{bmatrix}$	$\begin{bmatrix} -3.1351 \\ 2.2759 \end{bmatrix}$	$\begin{bmatrix} -3.9553 \\ 1.291 \end{bmatrix}$	$\begin{bmatrix} -4.0714 \\ 0.0888 \end{bmatrix}$	$\begin{bmatrix} -3.7561 \\ 1.1051 \end{bmatrix}$
Series	$\begin{bmatrix} -0.5634 \\ 2.3686 \end{bmatrix}$	$\begin{bmatrix} -1.8664 \\ 1.8863 \end{bmatrix}$	$\begin{bmatrix} -2.5352 \\ 0.9255 \end{bmatrix}$	$\begin{bmatrix} -2.7168 \\ -0.1767 \end{bmatrix}$	$\begin{bmatrix} -2.3905 \\ 0.7869 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -1.2968 \\ 2.6833 \end{bmatrix}$	$\begin{bmatrix} -3.1351 \\ 2.2759 \end{bmatrix}$	$\begin{bmatrix} -3.9553 \\ 1.291 \end{bmatrix}$	$\begin{bmatrix} -4.0714 \\ 0.0888 \end{bmatrix}$	$\begin{bmatrix} -3.7561 \\ 1.1051 \end{bmatrix}$
Template	$\begin{bmatrix} -2.0908 \\ 2.2071 \end{bmatrix}$	$\begin{bmatrix} -4.0409 \\ 2.8051 \end{bmatrix}$	$\begin{bmatrix} -4.5904 \\ 2.2904 \end{bmatrix}$	$\begin{bmatrix} -4.2069 \\ 1.3688 \end{bmatrix}$	$\begin{bmatrix} -3.3126 \\ 1.7582 \end{bmatrix}$

show the maximum likelihood estimates of configuration location. It is important to note that the exact configuration density is just a polynomial of 10 degree, which can be constructed by using the exact formulae given by Caro-Lopera *et al*(2007). Also, observe that all the estimates after this truncation are equal, as it can be noticed in the table. However, the estimates based on the truncation of the series given in lemma 3.1) are so unstable, in fact, it is not sufficient a truncation of 120 in the modification of the algorithms of Koev and Edelman (2006) to obtain the exact estimation. Moreover, the computations were performed with a processor Intel(R) Core(TM)2 Duo CPU, E7400@2.80GHz, and 2,96GB of RAM, and we tried to obtained an estimation with a truncation of 150, with the same initial values, and the programm does not tolerate that truncation. Tables also shows the configuration coordi-

Table 3: ...Continuation Table 1

	V_6	V_7	V_8	V_9	V_{10}
Series	$\begin{bmatrix} -0.2582 \\ 0.4178 \end{bmatrix}$	$\begin{bmatrix} 0.0624 \\ -0.3006 \end{bmatrix}$	$\begin{bmatrix} 0.117 \\ 0.0974 \end{bmatrix}$	$\begin{bmatrix} -0.1171 \\ 0.0412 \end{bmatrix}$	$\begin{bmatrix} -0.1083 \\ 0.5711 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -0.393 \\ 0.1695 \end{bmatrix}$	$\begin{bmatrix} -0.4416 \\ 0.2615 \end{bmatrix}$	$\begin{bmatrix} -0.6074 \\ 0.0661 \end{bmatrix}$	$\begin{bmatrix} -0.7698 \\ -0.3256 \end{bmatrix}$	$\begin{bmatrix} -0.822 \\ -0.6514 \end{bmatrix}$
Series	$\begin{bmatrix} -0.251 \\ 2.519 \end{bmatrix}$	$\begin{bmatrix} -0.5782 \\ 1.5909 \end{bmatrix}$	$\begin{bmatrix} -1.0868 \\ -0.1355 \end{bmatrix}$	$\begin{bmatrix} -1.0048 \\ -0.8559 \end{bmatrix}$	$\begin{bmatrix} -1.4068 \\ 0.3555 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -3.7978 \\ 1.2843 \end{bmatrix}$	$\begin{bmatrix} -4.5051 \\ 1.6567 \end{bmatrix}$	$\begin{bmatrix} -5.2428 \\ 1.3901 \end{bmatrix}$	$\begin{bmatrix} -5.3697 \\ 0.5387 \end{bmatrix}$	$\begin{bmatrix} -4.8461 \\ -0.3302 \end{bmatrix}$
Series	$\begin{bmatrix} -0.5192 \\ 0.5463 \end{bmatrix}$	$\begin{bmatrix} -0.5901 \\ 0.7407 \end{bmatrix}$	$\begin{bmatrix} -0.794 \\ 0.4857 \end{bmatrix}$	$\begin{bmatrix} -1.0118 \\ -0.1144 \end{bmatrix}$	$\begin{bmatrix} -1.0649 \\ -0.668 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -3.983 \\ 1.9115 \end{bmatrix}$	$\begin{bmatrix} -4.708 \\ 2.4794 \end{bmatrix}$	$\begin{bmatrix} -5.5607 \\ 2.0488 \end{bmatrix}$	$\begin{bmatrix} -5.8348 \\ 0.7249 \end{bmatrix}$	$\begin{bmatrix} -5.395 \\ -0.6145 \end{bmatrix}$
Series	$\begin{bmatrix} -0.7521 \\ 0.7489 \end{bmatrix}$	$\begin{bmatrix} -0.8495 \\ 0.9931 \end{bmatrix}$	$\begin{bmatrix} -1.1504 \\ 0.6829 \end{bmatrix}$	$\begin{bmatrix} -1.4295 \\ -0.101 \end{bmatrix}$	$\begin{bmatrix} -1.5055 \\ -0.8253 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -3.983 \\ 1.9115 \end{bmatrix}$	$\begin{bmatrix} -4.708 \\ 2.4794 \end{bmatrix}$	$\begin{bmatrix} -5.5607 \\ 2.0488 \end{bmatrix}$	$\begin{bmatrix} -5.8348 \\ 0.7249 \end{bmatrix}$	$\begin{bmatrix} -5.3955 \\ -0.6145 \end{bmatrix}$
Series	$\begin{bmatrix} -1.2986 \\ 1.0772 \end{bmatrix}$	$\begin{bmatrix} -1.493 \\ 1.4286 \end{bmatrix}$	$\begin{bmatrix} -1.9345 \\ 1.0284 \end{bmatrix}$	$\begin{bmatrix} -2.2848 \\ -0.0132 \end{bmatrix}$	$\begin{bmatrix} -2.318 \\ -0.9888 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -3.983 \\ 1.9115 \end{bmatrix}$	$\begin{bmatrix} -4.708 \\ 2.4794 \end{bmatrix}$	$\begin{bmatrix} -5.5607 \\ 2.0488 \end{bmatrix}$	$\begin{bmatrix} -5.8348 \\ 0.7249 \end{bmatrix}$	$\begin{bmatrix} -5.3955 \\ -0.6145 \end{bmatrix}$
Series	$\begin{bmatrix} -1.9591 \\ 1.3502 \end{bmatrix}$	$\begin{bmatrix} -2.2843 \\ 1.784 \end{bmatrix}$	$\begin{bmatrix} -2.8528 \\ 1.3471 \end{bmatrix}$	$\begin{bmatrix} -3.2177 \\ 0.1607 \end{bmatrix}$	$\begin{bmatrix} -3.1477 \\ -0.9718 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -3.983 \\ 1.9115 \end{bmatrix}$	$\begin{bmatrix} -4.708 \\ 2.4794 \end{bmatrix}$	$\begin{bmatrix} -5.5607 \\ 2.0488 \end{bmatrix}$	$\begin{bmatrix} -5.8348 \\ 0.7249 \end{bmatrix}$	$\begin{bmatrix} -5.3955 \\ -0.6145 \end{bmatrix}$
Series	$\begin{bmatrix} -2.1715 \\ 1.4227 \end{bmatrix}$	$\begin{bmatrix} -2.5401 \\ 1.8768 \end{bmatrix}$	$\begin{bmatrix} -3.1444 \\ 1.4358 \end{bmatrix}$	$\begin{bmatrix} -3.5064 \\ 0.2207 \end{bmatrix}$	$\begin{bmatrix} -3.3982 \\ -0.946 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -3.983 \\ 1.9115 \end{bmatrix}$	$\begin{bmatrix} -4.708 \\ 2.4794 \end{bmatrix}$	$\begin{bmatrix} -5.5607 \\ 2.0488 \end{bmatrix}$	$\begin{bmatrix} -5.8348 \\ 0.7249 \end{bmatrix}$	$\begin{bmatrix} -5.3955 \\ -0.6145 \end{bmatrix}$
Series	$\begin{bmatrix} -2.4511 \\ 1.5105 \end{bmatrix}$	$\begin{bmatrix} -2.8769 \\ 1.9884 \end{bmatrix}$	$\begin{bmatrix} -3.5264 \\ 1.5443 \end{bmatrix}$	$\begin{bmatrix} -3.8811 \\ 0.2992 \end{bmatrix}$	$\begin{bmatrix} -3.7205 \\ -0.9059 \end{bmatrix}$
Polynomial	$\begin{bmatrix} -3.983 \\ 1.9115 \end{bmatrix}$	$\begin{bmatrix} -4.708 \\ 2.4794 \end{bmatrix}$	$\begin{bmatrix} -5.5607 \\ 2.0488 \end{bmatrix}$	$\begin{bmatrix} -5.8348 \\ 0.7249 \end{bmatrix}$	$\begin{bmatrix} -5.3955 \\ -0.6145 \end{bmatrix}$
Template	$\begin{bmatrix} -3.5881 \\ 2.7053 \end{bmatrix}$	$\begin{bmatrix} -5.4996 \\ 4.0629 \end{bmatrix}$	$\begin{bmatrix} -7.5557 \\ 4.8428 \end{bmatrix}$	$\begin{bmatrix} -8.2514 \\ 4.4208 \end{bmatrix}$	$\begin{bmatrix} -6.9108 \\ 2.8899 \end{bmatrix}$

nates of the template digit 3, a figure consisting of two equal sized arcs, and 13 landmarks (two coincident) lying on two regular octagons see Dryden and Mardia(1998), p.153. Finally a test about the equality of the configuration shape and the templates indicates an approximate zero p-value, the same conclusion obtained by Dryden and Mardia(1998) and Caro-Lopera *et al*(2009) by using other approaches.

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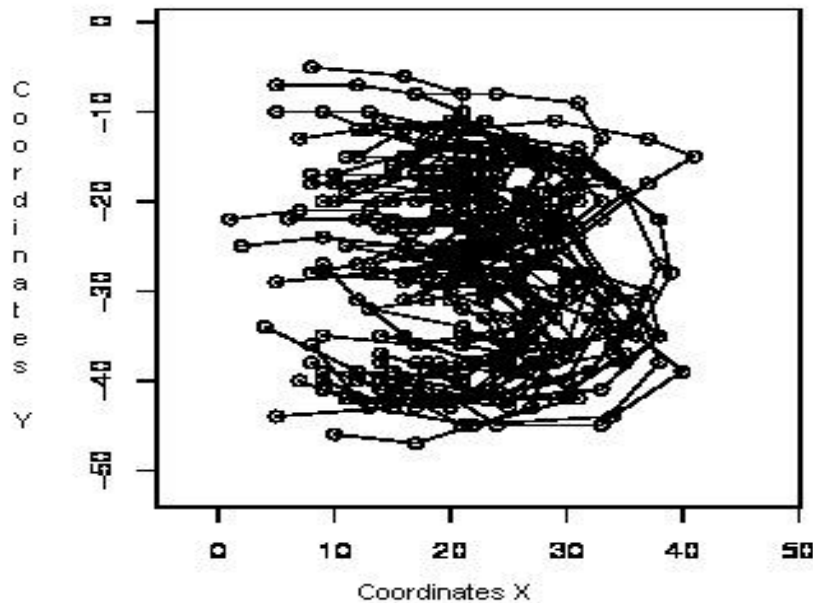


Figure 1: Sample of 30 handwritten digit 3.

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