

## Rosenthal's Type Inequalities for Negatively Orthant Dependent Random Variables

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**Abstract.** In this paper, we obtain some Rosenthal's type inequalities for negatively orthant dependent (NOD) random variables.

### 1 Introduction

Let  $\{X_n, n \geq 1\}$  be a sequence of centered and independent random variables. Put  $S_n = X_1 + \cdots + X_n$ . Suppose that for all  $n \geq 1$ ,  $EX_n^p < \infty$ . Rosenthal's inequality (cf. Petrov [4]) yields the existence of a positive constant  $C_p$  that depends only on  $p$ , for which,

$$E|S_n|^p \leq C_p \left( \sum_{k=1}^n E|X_k|^p + (Var S_n)^{p/2} \right), \quad p \geq 2.$$

Inequalities of this kind are very important since they reduce (for  $n$  sufficiently large) the behaviors of  $E|S_n|^p$  to those of  $(Var S_n)^{p/2}$ . Their main interest is that they give the right bound for integrated moments in non-parametric estimation (cf. Doukhan [2] for more

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about the interest of those inequalities in probability and statistics.) Our purpose in this paper is to give versions of such inequalities for negatively orthant dependent random variables. The definition of which is as follows.

**Definition 1.1.** A finite family of random variables is said to be negatively orthant dependent (NOD) if for all real  $x_1, \dots, x_n$

$$P(X_1 > x_1, \dots, X_n > x_n) \leq \prod_{i=1}^n P(X_i > x_i),$$

and

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) \leq \prod_{i=1}^n P(X_i \leq x_i).$$

An infinite family of random variables is NOD if every finite subfamily is NOD. More about negatively dependent random variables is found in Bozorgnia et al. [1].

The paper is organized as follows. In Section 2, we give our main results. In Section 3, we prove the main results.

## 2 Main results

In this section, we give some versions of Rosenthal's type inequalities for sequences of negatively orthant dependent random variables.

Throughout this note, we shall suppose that  $X_1, \dots, X_n$  is a finite family of NOD random variables with respective distribution functions  $F_1, \dots, F_n$ . Set

$$M_{s,n} = \sum_{k=1}^n E|X_k|^s \quad \text{and} \quad S_n = \sum_{k=1}^n EX_k.$$

**Theorem 2.1.** *Let  $0 < t \leq 1$  and  $g(x)$  be a non-negative even function, non-decreasing on the positive half-line, and satisfy the condition  $g(0) = 0$ . Let  $Eg(X_k) < \infty, k = 1, 2, \dots, n$ , then for every  $r > 0$ ,*

$$Eg(S_n) \leq \sum_{k=1}^n Eg(rX_k) + 2e^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x). \quad (2.1)$$

□

**Theorem 2.2.** Let  $1 \leq t \leq 2$  and  $EX_k = 0$ ,  $k = 1, 2, \dots, n$ , under the assumptions of Theorem 2.1,

$$Eg(S_n) \leq \sum_{k=1}^n Eg(rX_k) + 2e^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x). \quad (2.2)$$

□

**Corollary 2.1.** Let  $0 < t \leq 1$ ,  $p \geq t$ . Then

$$E|S_n|^p \leq C(p, t)(M_{p,n} + M_{t,n}^{p/t}), \quad (2.3)$$

where,  $C(p, t)$  is a positive constant depending on  $p$  and  $t$ . □

**Corollary 2.2.** Let  $1 \leq t \leq 2$ ,  $p \geq t$ . If  $EX_k = 0$ ,  $k = 1, \dots, n$ , then

$$E|S_n|^p \leq C(p, t)(M_{p,n} + M_{t,n}^{p/t}). \quad (2.4)$$

□

**Corollary 2.3.** Let  $0 < t \leq 1$ ,  $p \geq t$ , then

$$\begin{aligned} E|S_n|^p &\leq C(1 + n^{\frac{p}{t}-1})M_{p,n} \\ &\leq 2Cn^{\frac{p}{t}-1}M_{p,n}, \end{aligned} \quad (2.5)$$

and if

$$n^{-1} \sum_{k=1}^n P(X_k \neq 0) < 1,$$

then

$$E|S_n|^p \leq C(1 + [\sum_{k=1}^n P(X_k \neq 0)]^{\frac{p}{t}-1})M_{p,n}, \quad (2.6)$$

where  $C = C(p, t)$ . □

**Corollary 2.4.** Let  $1 \leq t \leq 2$ ,  $p \geq t$  and  $EX_k = 0$ ,  $k = 1, \dots, n$ .

i) 
$$\begin{aligned} E|S_n|^p &\leq C(1 + n^{\frac{p}{t}-1})M_{p,n} \\ &\leq 2Cn^{\frac{p}{t}-1}M_{p,n} \end{aligned} \quad (2.7)$$

ii) If  $n^{-1} \sum_{k=1}^n P(X_k \neq 0) < 1$ , then

$$E|S_n|^p \leq C(1 + [\sum_{k=1}^n P(X_k \neq 0)]^{\frac{p}{t}-1})M_{p,n}. \quad (2.8)$$

□

**Remark.** Rivaz et al. [5] obtained some moment inequalities for NOD random variables. Theorem 3 and Corollary 3 in mentioned article are results of Corollary 2.1 with  $t = 1$  (without condition  $EX_n = 0, n \geq 1$ ) and Corollary 2.2 with  $t = 2$ , respectively.

### 3 Proofs

Proofs are based on the following lemmas.

**Lemma 3.1.** (see Fakoor and Azarnoosh [3], Theorem 3) *Let  $0 < t \leq 1$ . Then for any  $h, x, y > 0$*

$$P(|S_n| \geq x) \leq \sum_{k=1}^n P(|X_k| \geq y) + 2 \exp\left\{\frac{x}{y} - \frac{x}{y} \log\left(1 + \frac{xy^{t-1}}{M_{t,n}}\right)\right\}, \quad (3.1)$$

where

$$M_{t,n} = \sum_{k=1}^n E|X_k|^t.$$

□

It is easy to see that with some changes in the proof of Theorem 3 in Fakoor and Azarnoosh [3], we have the following lemma,

**Lemma 3.2.** *Let  $1 \leq t \leq 2$ . If  $EX_k = 0, k = 1, 2, \dots, n$ , then for any  $h, x, y > 0$*

$$P(|S_n| \geq x) \leq \sum_{k=1}^n P(|X_k| \geq y) + 2 \exp\left\{\frac{e^{hy} - 1 - hy}{y^t} M_{t,n} - hx\right\}. \quad (3.2)$$

□

Proof of the main results are the following.

**Proof of Theorem 2.1.** Put  $\frac{x}{y} = r$ , in (3.1), we have

$$P(|S_n| \geq x) \leq \sum_{k=1}^n P(|X_k| \geq \frac{x}{r}) + 2e^r \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r},$$

then

$$\int_0^\infty P(|S_n| \geq x) dg(x) \leq \sum_{k=1}^n \int_0^\infty P(|X_k| \geq \frac{x}{r}) dg(x) + 2e^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x).$$

Now by Lemma 2.4. in Petrov [4], we have

$$Eg(S_n) \leq \sum_{k=1}^n Eg(rX_k) + 2e^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x).$$

This complete the proof. □

**Proof of Theorem 2.2.** We set

$$h = \frac{1}{y} \log\left(1 + \frac{xy^{t-1}}{M_{t,n}}\right),$$

in the right hand side of (3.2). Since

$$\frac{M_{t,n}}{y^t} \log\left(1 + \frac{xy^{t-1}}{M_{t,n}}\right) \geq 0, \tag{3.3}$$

therefore we have,

$$P(|S_n| \geq x) \leq \sum_{k=1}^n P(|X_k| \geq y) + 2 \exp\left\{\frac{x}{y} - \frac{x}{y} \log\left(1 + \frac{xy^{t-1}}{M_{t,n}}\right)\right\}. \tag{3.4}$$

Put  $\frac{x}{y} = r$ , in (3.4),

$$P(|S_n| \geq x) \leq \sum_{k=1}^n P(|X_k| \geq \frac{x}{r}) + 2e^r \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r},$$

then

$$\int_0^\infty P(|S_n| \geq x) dg(x) \leq \sum_{k=1}^n \int_0^\infty P(|X_k| \geq \frac{x}{r}) dg(x) + 2e^r \int_0^\infty \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dg(x). \tag{3.5}$$

Now, by Lemma 2.4 in Petrov [4], complete the proof. □

**Proof of Corollary 2.1.** By putting  $g(x) = |x|^p$  in Theorem 2.1, then for  $p \geq t$ ,

$$E|S_n|^p \leq r^p \sum_{k=1}^n E|X_k|^p + 2pe^r \int_0^\infty x^{p-1} \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dx. \tag{3.6}$$

Let

$$I = \int_0^\infty x^{p-1} \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dx.$$

It is easy to see that, for  $r > p/t$

$$I = \frac{B(\frac{p}{t}, r - \frac{p}{t})}{t} r^{\frac{t-1}{t}p} M_{t,p}^{\frac{p}{t}},$$

where

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \quad \alpha, \beta > 0$$

is the Beta function. Substitute  $I$  in (3.6) and choose  $C(p, t) = \max\{r^p, 2pe^r \frac{B(\frac{p}{t}, r - \frac{p}{t})}{t} r^{\frac{t-1}{t}p}\}$ , we obtain the result.  $\square$

**Proof of Corollary 2.2.** In Theorem 2.2, let  $g(x) = |x|^p, p \geq t$ , then

$$E|S_n|^p \leq r^p \sum_{k=1}^n E|X_k|^p + 2pe^r \int_0^\infty x^{p-1} \left(1 + \frac{x^t}{r^{t-1}M_{t,n}}\right)^{-r} dx.$$

Hence for  $r > p/t$ ,

$$E|S_n|^p \leq r^p \sum_{k=1}^n E|X_k|^p + 2pe^r \frac{B(\frac{p}{t}, r - \frac{p}{t})}{t} r^{\frac{t-1}{t}p} \left(\sum_{k=1}^n E|X_k|^t\right)^{\frac{p}{t}}.$$

With  $C(p, t)$  as the proof of Corollary 2.1. This complete the proof.  $\square$

**Proof of Corollary 2.3.** Let  $X_1, X_2, \dots, X_n$  are NOD random variables with respective distribution functions  $F_1, F_2, \dots, F_n$  and  $Y$  be a random variable with the distribution function  $n^{-1} \sum_{k=1}^n F_k(x)$ . It is easy to see that, for  $r > 0$

$$E|Y|^r = n^{-1} \sum_{k=1}^n E|X_k|^r,$$

and

$$P(Y \neq 0) = n^{-1} \sum_{k=1}^n P(X_k \neq 0).$$

Applying Lyapunov's inequality (cf. petrov [4], page 62), we have

$$M_{t,n}^{p/t} \leq n^{p/t-1} M_{p,n}. \quad (3.7)$$

If

$$n^{-1} \sum_{k=1}^n P(X_k \neq 0) < 1,$$

then by improvement Lyapunov's inequality,

$$M_{t,n}^{p/t} \leq \left[ \sum_{k=1}^n P(X_k \neq 0) \right]^{\frac{p}{t}-1} M_{p,n}. \quad (3.8)$$

By applying (3.7) and (3.8) to the right hand side of (2.3), in Corollary 2.1, we have (2.5) and (2.6).  $\square$

**Proof of Corollary 2.4.** By applying (3.7) and (3.8) to the right hand side of (2.4), we obtain (2.7) and (2.8).  $\square$

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