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Almost Sure Convergence Rates for the Estimation of a Covariance Operator for Negatively Associated Samples

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Abstract. Let $\{X_n,\ n \geq 1\}$ be a strictly stationary sequence of negatively associated random variables, with common continuous and bounded distribution function F. In this paper, we consider the estimation of the two-dimensional distribution function of (X_1, X_{k+1}) based on histogram type estimators as well as the estimation of the covariance function of the limit empirical process induced by the sequence $\{X_n,\ n \geq 1\}$. Then, we derive uniform strong convergence rates for two-dimensional distribution function of (X_1, X_{k+1}) without any condition on the covariance structure of the variables. Finally, assuming a convenient decrease rate of the covariances

$$Cov(X_1, X_{n+1}),\ n \geq 1,$$

we introduce uniform strong convergence rate for covariance function of the limit empirical process.

Key words and phrases: Empirical process, histogram estimator, negative association, stationarity.

1 Introduction, definitions and assumption

Let $Z(t) = n^{-1/2}\sum_{i=1}^{n}(1_{(-\infty,t]}(X_i) - F(t))$ be the empirical process induced by the random variables $\{X_n,\ n \geq 1\}$, with common continuous distribution function F, where 1_A represents the indicator function of the set A. As it is well known, the limit behavior of the empirical process has been intensively studied in recent years due to the importance of this function to many statistical applications. In several fields of statistics we often find transformations of the empirical process for which it is of interest to characterize their limit in distribution. The results about the asymptotic behavior of the empirical process are a valuable tool to accomplish this. Some classic examples are several goodness of fit tests statistics, such as the Kolmogorov-Smirnov and the Cramer-von Mises ω^2 test statistics, which are, respectively, the sup-norm and the $L^2[0,1]$ norm of the uniform empirical process. Another example of application may be found in Shao and Yu [14], who are interested on integral functionals of the empirical process and on the mean residual life process in reliability. It is well known that the study of convergence of $Z(t)$ can be carried out supposing the variables $\{X_n,\ n \geq 1\}$ to be uniformly distributed on $[0,1]$. This is the uniform empirical process. For independent random variables, the uniform empirical process converges in the Skorohod space $D[0,1]$ to the Brownian bridge, a centered Gaussian process with covariance function $\Gamma(r,s) = F_k(r,s) - F(r)F(s)$ where $F_k(r,s)$ is the distribution function of (X_1, X_{k+1}). For dependent sequences, under certain conditions (see Newman [10] Theorem 17 and the first remark of p. 137), the limit of the uniform empirical process still is a centered Gaussian process, but the covariance function changes to

$$\begin{aligned}\Gamma(r,s) = F_k(r,s) - F(r)F(s) \\ &+ \sum_{k=1}^{\infty}(P_r(X_1 \leq r, X_{k+1} \leq s) - F(r)F(s)) \\ &+ \sum_{k=1}^{\infty}(P_r(X_1 \leq s, X_{k+1} \leq r) - F(s)F(r)). \quad (1)\end{aligned}$$

Henriques and Oliveira [3] proved strong convergence rates for the estimation of Γ for positively associated random variables under some assumptions on the covariance structure of the variables. In this article, we consider negative association and prove strong convergence rates for the estimation of Γ that are different from results of Henriques and Oliveira [3].

A finite family of random variables $\{X_i,\ 1 \le i \le n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, ..., n\}$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \le 0$$

whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA. This definition is introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [4]. Because of their wide applications in multivariate statistical analysis and reliability theory, the notion of NA has received more and more attention recently. We refer to Joag-Dev and Proschan [4] for fundamental properties, Newman [9] and Su and Chi [15] for central limit theorem, Matula [7] for three series theorem, Su et al. [16] for a moment inequality, a weak invariance principle and example to show that there exists an infinite family of non-degenerate non-independent strictly stationary NA random variables, Shao [13] for the Rosenthal type maximal inequality and Kolmogorov exponenential inequality, Liang and Su [5] for convergence rates of law of the logarithm, Roussas [11] for the central limit theorem of random fields, some examples and applications and Yuan et al. [17] for improving the result of Roussas [11].

The above comments motivated the interest on the estimation of the covariance function (1). For this we will estimate the terms appearing in the series and sum a convenient number of these estimates to approximate Γ. We will concentrate on histogram estimators and on proving uniform strong convergence rates.

The estimator for $F_k(r, s)$ is defined by

$$\hat{F}_k(r, s) = \frac{1}{n-k} \sum_{i=1}^{n-k} (1_{(-\infty, r]}(X_i) 1_{(-\infty, s]}(X_{i+k})). \tag{2}$$

Combining the estimator $\hat{F}_k(r, s)$ with the empirical distribution function defined by $\hat{F}(r) = n^{-1} \sum_{i=1}^{n} 1_{(-\infty, r]}(X_i)$, we obtain a natural estimator for the terms $\varphi_k(r, s) = F_k(r, s) - F(r)F(s)$, namely,

$$\hat{\varphi}_k(r, s) = \hat{F}_k(r, s) - \hat{F}(r)\hat{F}(s). \tag{3}$$

The estimators for the infinite sum in the expression of $\Gamma(r, s)$ and for $\Gamma(r, s)$ itself are, respectively,

$$\sum_{k=1}^{a_n} \hat{\varphi}_k(r, s), \tag{4}$$

and

$$\hat{\Gamma}(r, s) = \hat{F}_k(r, s) - \hat{F}(r)\hat{F}(s) + \sum_{k=1}^{a_n} (\hat{\varphi}_k(r, s) + \hat{\varphi}_k(s, r)), \tag{5}$$

where $a_n \to +\infty$ is such that $\frac{a_n}{n} \to 0$ as $n \to \infty$.

We now introduce a general assumption to be used throughout the article.

(A). $\{X_n, \ n \geq 1\}$, is an NA and strictly stationary sequence of random variables having density function bounded by M_0; let $M_1 = 2\max(2/\pi^2, 45M_0)$.

In this study, we derive uniform strong convergence rates for two-dimensional distribution function of (X_1, X_{k+1}) and $\varphi_k(r, s)$ which approach the best possible ones for independent samples and do not need any condition on the covariance structure of the variables. Furthermore, for estimation of covariance function of the limit empirical process, we obtain convergence rates assuming a convenient decrease rate of the covariances $Cov(X_1, X_{n+1}), \ n \geq 1$. For strictly stationary sequence of NA random variables, $\sum_{j=2}^{\infty} Cov(X_1, X_j)$ is absolutely convergent (Lemma 8 in Newman [10]). Hence, it is clear under Assumption (A) $Cov(X_1, X_{n+1}), \ n \geq 1$ exists. The starting point for the derivation of these rates is a moment inequality for NA random variables by Su et al. [16] and Matula [8]. Our method was inspired by Masry [6], who uses moment inequalities to obtain convergence rates for the estimation of the density and its derivatives, but considering NA samples.

In Section 2 we will present some auxiliary results needed to establish the above mentioned convergence rates. The moment inequality referred earlier is included in this section. The results establishing rates of uniform strong convergence are presented in Sections 3 and 4, Section 3 deals with the estimators $\hat{F}_k(r, s)$ and $\hat{\varphi}_k(r, s)$ and Section 4 with the estimators $\sum_{k=1}^{a_n} \hat{\varphi}_k(r, s)$ and $\hat{\Gamma}(r, s)$. Section 5 summarizes our results.

2 Auxiliary results

In this section we introduce a moment inequality for NA random variables and an inequality that are needed for proving our convergence rates. Throughout this paper the letter C stands for a positive constant, which may take different values at each appearance. In each case the value of the constant is independent of n but may depend on k and p.

Lemma 2.1. [Su et al. [16] and Matula [8]] *Let $(X_1, X_2, ..., X_n)$ be an NA random vector with $EX_j = 0$ and $E|X_j|^p < \infty$ for some $p \geq 2$ and all $j = 1, ..., n$. Then there exists a constant $C = C(p) > 0$, such that*

$$E|\sum_{j=1}^{n} X_j|^p \leq C[\sum_{j=1}^{n} E|X_j|^p + (\sum_{j=1}^{n} EX_j^2)^{p/2}]. \quad \square \tag{6}$$

Based on the previous result we prove an inequality that will be essential for proving our convergence rates.

Lemma 2.2. *Let $k \in I\!N_0$ be fixed and ε_n a sequence of positive numbers. Suppose (A) is satisfied. Then, there exists a constant $C = C(p)$ such that, for each $n > k$, $p > 2$ and $r, s \in I\!R$,*

$$P_r(|\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) \leq \frac{C}{\varepsilon_n^p (n-k)^{p/2}}. \tag{7}$$

Proof. For each $n \in I\!N$ and fixed $r, s \in I\!R$ define

$$W_{k,n} = 1_{(-\infty,r]}(X_n) 1_{(-\infty,s]}(X_{k+n}) - F_k(r,s),$$

so that we can write

$$\hat{F}_k(r,s) - F_k(r,s) = \frac{1}{n-k} \sum_{i=1}^{n-k} W_{k,i}.$$

Given (A), since the $W_{k,n}$ are decreasing functions of the variables X_n, the sequence $\{W_{k,n},\ n \geq 1\}$, is NA and strictly stationary. Furthermore, $|W_{k,n}| \leq 1$ and $E(W_{k,n}) = 0$ then, $E|W_{k,n}|^p < \infty$, for each $n \geq 1$ and $p > 2$.

We want to apply Lemma 2.1 to the sequence $\{W_{k,n},\ n \geq 1\}$. So there exists a constant $C = C(p)$ such that, for all $n \geq 1$

$$E|\sum_{i=1}^{n} W_{k,i}|^p \quad \leq \quad C[\sum_{i=1}^{n} E|W_{k,i}|^p + (\sum_{i=1}^{n} EW_{k,i}^2)^{p/2}]$$

$$\begin{aligned} &\leq C[n + n^{p/2}] \\ &\leq Cn^{p/2}. \end{aligned} \tag{8}$$

Using the Markov inequality we find, for all $n > k$,

$$\begin{aligned} P_r(|\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) &\leq \frac{1}{\varepsilon_n^p (n-k)^p} E|\sum_{i=1}^{n-k} W_{k,i}|^p \\ &\leq \frac{C}{\varepsilon_n^p (n-k)^{p/2}}. \quad \square \end{aligned} \tag{9}$$

For the formulation of the next results we need to introduce some additional notation. Let t_n be a sequence of positive integers such that $t_n \to +\infty$. For each $n \in I\!N$ and each $i = 1, ..., t_n$, put $x_{n,i} = Q(i/t_n)$, where Q is the quantile function of F. Define then, for $n, k \in I\!N$,

$$D_{n,k} = \sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)|,$$

and

$$D^*_{n,k} = \max_{i,j=1,\ldots,t_n} |\hat{F}_k(x_{n,i}, x_{n,j}) - F_k(x_{n,i}, x_{n,j})|.$$

To prove an uniform version of the preceding lemma we will apply the following result which is proved in Theorem 2 of Henriques and Oliveira [2].

Lemma 2.3. *If the sequence $\{X_n,\ n \geq 1\}$ satisfies (A) then, for each $n \in I\!N$ and each $k \in I\!N_0$,*

$$D_{n,k} \leq D^*_{n,k} + \frac{2}{t_n} \qquad a.s.\ .\ \square \tag{10}$$

Lemma 2.4. *Let ε_n and t_n be two sequences of positive numbers such that $t_n \to +\infty$ and $\varepsilon_n t_n \to +\infty$, and $k \in I\!N_0$ be fixed. Suppose (A) holds. Then, for some $p > 2$ and any large enough n,*

$$P_r(\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) \leq \frac{2^p t_n^2 C}{\varepsilon_n^p (n-k)^{p/2}}. \tag{11}$$

Proof. From Lemma 2.3 we obtain

$$\begin{aligned} P_r(D_{n,k} > \varepsilon_n) &\leq P_r(D^*_{n,k} + \frac{2}{t_n} > \varepsilon_n) \\ &\leq P_r(D^*_{n,k} > \frac{\varepsilon_n}{2}) + P_r(\frac{2}{t_n} > \frac{\varepsilon_n}{2}). \end{aligned}$$

Since $\varepsilon_n t_n \to +\infty$ as $n \to +\infty$, we conclude $P_r(\frac{2}{t_n} > \frac{\varepsilon_n}{2}) \to 0$ as $n \to +\infty$. So, there exists an $n_0 \in I\!N$ such that for every $n \geq n_0$

$$\begin{aligned} & P_r(D_{n,k} > \varepsilon_n) \\ & \quad \leq \sum_{i,j=1,\ldots,t_n} P_r(|\hat{F}_k(x_{n,i}, x_{n,j}) - F_k(x_{n,i}, x_{n,j})| > \frac{\varepsilon_n}{2}) \\ & \quad \leq t_n^2 \max_{i,j=1,\ldots,t_n} P_r(|\hat{F}_k(x_{n,i}, x_{n,j}) - F_k(x_{n,i}, x_{n,j})| > \frac{\varepsilon_n}{2}). \end{aligned} \tag{12}$$

Now, apply Lemma 2.2 to complete the proof. □

3 Uniform strong convergence rates for $\hat{F}_k$

In this section, we use the results of last section to establish uniform strong convergence rates for the estimator $\hat{F}_k$.

Lemma 3.1. *Let $k \in I\!N_0$ be fixed and suppose (A) holds. Then, for some $p > 2$ and every $0 < \delta < \frac{p-2}{2}$, we have*

$$\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| = O((log\ n)^{\frac{2}{p+2}} n^{-\frac{p-2-2\delta}{2p+4}}) \qquad a.s.\ . \tag{13}$$

Proof. Fix $0 < \delta < \frac{p-2}{2}$ and put $t_n = \frac{log\ n}{\varepsilon_n}$ in order to have $\varepsilon_n t_n \to +\infty$. Now, choosing $\varepsilon_n = (log\ n)^{\frac{2}{p+2}} n^{-\frac{p-2-2\delta}{2p+4}}$, we will obtain from Lemma 2.4 for n large enough,

$$P_r(\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| > \varepsilon_n) \leq C \frac{(log\ n)^2}{\varepsilon_n^{p+2}(n-k)^{p/2}} \leq C n^{-(1+\delta)}.$$

The sequence on the right-hand side above being summable, the result follows by the Borel-Cantelli Lemma. □

Note that, $\frac{p-2-2\delta}{2p+4}$ approaches $1/2$ as p grows to ∞, so the convergence rate established in the previous lemma can be arbitrarily close to $n^{-1/2}$, if a sufficiently large p can be chosen. As stated in the next theorem, this is always possible.

Theorem 3.1. *Let $k \in I\!N_0$ be fixed and suppose (A) holds. Then we have, for every $0 < \gamma < 1/2$,*

$$\sup_{r,s \in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| = O(n^{-\gamma}) \qquad a.s.\ . \tag{14}$$

Proof. Fix $0 < \gamma < 1/2$. Now, choose $p > 2$ and $0 < \delta < p/2 - 1$ so that $\frac{p-2-2\delta}{2p+4} > \gamma$. From Lemma 3.1, it follows that

$$\sup_{r,s\in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| \leq C(log\ n)^{\frac{2}{p+2}} n^{-\frac{p-2-2\delta}{2p+4}} \leq Cn^{-\gamma} \qquad a.s.\ . \ \square$$

Note that with $k = 0$ and $r = s$ the estimator $\hat{F}_k(r,s)$ reduces to the one-dimensional empirical distribution function $\hat{F}(s)$. So, the results of the previous theorem stay valid for $\hat{F}$. In fact, under the condition of Lemma 2.4, with $k = 0$, we would obtain, for every n large enough,

$$P_r(\sup_{s\in I\!R} |\hat{F}(s) - F(s)| > \varepsilon_n) \leq \frac{2^p t_n C}{\varepsilon_n^p n^{p/2}}.$$

Then, as it is displayed in the proofs of Theorem 3.1, we would find that, for every $0 < \gamma < 1/2$,

$$\sup_{s\in I\!R} |\hat{F}(s) - F(s)| = O(n^{-\gamma}) \qquad a.s.\ . \tag{15}$$

We note also that the convergence rate is near the optimal rate for $\hat{F}$, in the independent setting. In fact, for independent samples, the Law of the Iterated Logarithm implies that the best possible convergence rate for the one-dimensional empirical distribution function is $O((\frac{log\ log\ n}{n})^{1/2})$, which is just slightly faster than the rate given in the previous theorem.

The next theorem is the analogue of Theorem 3.1 for the estimator $\hat{\varphi}_k$.

Theorem 3.2. *Let $k \in I\!N_0$ be fixed and suppose (A) holds. Then we have, for every $0 < \gamma < 1/2$,*

$$\sup_{r,s\in I\!R} |\hat{\varphi}_k(r,s) - \varphi_k(r,s)| = O(n^{-\gamma}) \qquad a.s.\ . \tag{16}$$

Proof. As

$$\begin{aligned}
&\sup_{r,s\in I\!R} |\hat{\varphi}_k(r,s) - \varphi_k(r,s)| \\
&\quad\leq \sup_{r,s\in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| + \sup_{r,s\in I\!R} |F(r)F(s) - \hat{F}(r)\hat{F}(s)| \\
&\quad\leq \sup_{r,s\in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| + \sup_{r,s\in I\!R} F(r)|F(s) - \hat{F}(s)| \\
&\qquad\qquad + \sup_{r,s\in I\!R} \hat{F}(s)|F(r) - \hat{F}(r)| \\
&\quad\leq \sup_{r,s\in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| + 2\sup_{s\in I\!R} |F(s) - \hat{F}(s)|, \qquad (17)
\end{aligned}$$

the result follows immediately from Theorem 3.1. □

4 Uniform strong convergence rates for $\hat{\Gamma}$

In this section we will derive uniform strong convergence rates for the estimators of the sum $\sum_{k=1}^{\infty} \varphi_k(r,s)$ and of the covariance function $\Gamma(r,s)$.

It is well known that the covariance structure of a sequence of NA random variables highly determines its approximate independence (see, for example, Newman [10] for a number of results regarding this). As a natural consequence, when dealing with NA samples it is common to have assumptions on the covariance structure of the random variables. To introduce the condition to be considered in the remained results of this article we define

$$u(n) = \sum_{j=n+1}^{\infty} |Cov^{1/3}(X_1, X_j)|. \tag{18}$$

Now, in order to be able to identify explicit convergence rates, we consider some conditions on the covariance structure of the random variables that are introduced in the next lemma.

The following lemma provides uniform strong convergence rate for the sum $\sum_{k=1}^{\infty} \varphi_k(r,s)$.

Lemma 4.1. *Let (A) holds and $\theta > 0$. Suppose $a_n = n^{\frac{p-2-2\delta}{p^2+3p}}$ for some $p > 2$ and for each $0 < \delta < \frac{p-2}{2}$. If*

$$u(n) \leq Cn^{-\theta}, \tag{19}$$

for all $n \geq 1$, we have

$$\sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} \hat{\varphi}_k(r,s) - \sum_{k=1}^{\infty} \varphi_k(r,s)| = O((log\, n)^{\frac{2}{p+2}} n^{-\frac{(p-2)(p-2-2\delta)}{2p(p+2)}}) \quad a.s.. \tag{20}$$

Proof. Let $0 < \delta < \frac{p-2}{2}$ and take $\varepsilon_n = (log\, n)^{\frac{2}{p+2}} n^{-\frac{(p-2)(p-2-2\delta)}{2p(p+2)}}$ and $t_n = \frac{a_n}{\varepsilon_n} log\, n$. Now, write

$$P_r(\sup_{r,s\in I\!R} |\sum_{k=1}^{a_n} (\hat{F}_k(r,s) - F_k(r,s))| > \varepsilon_n) \leq \sum_{k=1}^{a_n} P_r(\sup_{r,s\in I\!R} |\hat{F}_k(r,s) - F_k(r,s)| > \frac{\varepsilon_n}{a_n}). \tag{21}$$

Note that, as $0<\delta<\frac{p-2}{2}$, we have $\frac{(p-2)(p-2-2\delta)}{2p(p+2)}>0$ and $\frac{p-2-2\delta}{p^2+3p}>0$, so that $\varepsilon_n \to 0$, $a_n \to +\infty$, $t_n \to +\infty$ and $\frac{\varepsilon_n}{a_n}t_n \to +\infty$. Also, as $\frac{p-2-2\delta}{p^2+3p}<1$, $\frac{a_n}{n}\to 0$.

From (21), applying Lemma 2.4 with $\frac{\varepsilon_n}{a_n}$ replacing ε_n, we conclude that there exists a constant $C=C(p)$ such that, for all n large enough,

$$\begin{aligned} P_r(\sup_{r,s\in I\!R}|\sum_{k=1}^{a_n}(\hat{F}_k(r,s)-F_k(r,s))|>\varepsilon_n) & \\ \leq & \sum_{k=1}^{a_n}\frac{C2^p t_n^2 a_n^p}{\varepsilon_n^p(n-k)^{p/2}} \\ \leq & C\frac{2^p t_n^2 a_n^{p+1}}{\varepsilon_n^p(n-a_n)^{p/2}} \\ = & C\frac{a_n^{p+3}(log\ n)^2}{\varepsilon_n^{p+2}(n-a_n)^{p/2}}. \end{aligned} \tag{22}$$

By elementary manipulations it is easy to check that

$$\frac{(p-2)(p-2-2\delta)}{2p(p+2)}=\frac{p-2-2\delta}{p^2+3p}.\frac{p+3}{p+2}-\frac{p-2-2\delta}{2p+4}, \tag{23}$$

so, we may write $\varepsilon_n=(log\ n)^{\frac{2}{p+2}}a_n^{\frac{p+3}{p+2}}n^{-\frac{p-2-2\delta}{2p+4}}$. Inserting this on the right-hand side of (22) it follows that

$$P_r(\sup_{r,s\in I\!R}|\sum_{k=1}^{a_n}(\hat{F}_k(r,s)-F_k(r,s))|>\varepsilon_n)\leq C\frac{n^{\frac{p-2-2\delta}{2}}}{(n-a_n)^{p/2}}. \tag{24}$$

As $\frac{a_n}{n}\to 0$, we have $\frac{n^{\frac{p-2-2\delta}{2}}}{(n-a_n)^{p/2}}\sim n^{-(1+\delta)}$, thus the sequence on the upper bound of (24) is summable. Then, from Borel-Cantelli Lemma it follows that

$$\sup_{r,s\in I\!R}|\sum_{k=1}^{a_n}(\hat{F}_k(r,s)-F_k(r,s))|=O((log\ n)^{\frac{2}{p+2}}n^{-\frac{(p-2)(p-2-2\delta)}{2p(p+2)}})\qquad a.s.\ . \tag{25}$$

Now, we may write

$$|\sum_{k=1}^{a_n}\hat{\varphi}_k(r,s)-\sum_{k=1}^{\infty}\varphi_k(r,s)|$$

$$
\begin{aligned}
&\leq \ |\sum_{k=1}^{a_n}(\hat{\varphi}_k(r,s)-\varphi_k(r,s))| + |\sum_{k=a_n+1}^{\infty}\varphi_k(r,s)| \\
&\leq \ |\sum_{k=1}^{a_n}(\hat{F}_k(r,s)-F_k(r,s))| + a_n|F(s)-\hat{F}(s)| \\
&\quad +a_n|F(r)-\hat{F}(r)| + |\sum_{k=a_n+1}^{\infty}\varphi_k(r,s)|. \qquad (26)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sup_{r,s\in I\!R}|\sum_{k=1}^{a_n}\hat{\varphi}_k(r,s) - \sum_{k=1}^{\infty}\varphi_k(r,s)| \\
\leq \ \sup_{r,s\in I\!R}|\sum_{k=1}^{a_n}(\hat{F}_k(r,s)-F_k(r,s))| \\
+2a_n\sup_{s\in I\!R}|F(s)-\hat{F}(s)| \\
+\sup_{r,s\in I\!R}|\sum_{k=a_n+1}^{\infty}\varphi_k(r,s)|. \qquad (27)
\end{aligned}
$$

The convergence rate of the first term on the right-hand side of (27) is given in (25). From Lemma 3.1 the second term is almost surely $O(a_n(log\ n)^{\frac{2}{p+2}}n^{-\frac{p-2-2\delta}{2p+4}})$. Since $\frac{p+3}{p+2} > 1$ and taking into account (23), we have

$$
\begin{aligned}
a_n(log\ n)^{\frac{2}{p+2}}n^{-\frac{p-2-2\delta}{2p+4}} &< a_n^{\frac{p+3}{p+2}}(log\ n)^{\frac{2}{p+2}}n^{-\frac{p-2-2\delta}{2p+4}} \\
&= (log\ n)^{\frac{2}{p+2}}n^{-\frac{(p-2)(p-2-2\delta)}{2p(p+2)}}.
\end{aligned}
$$

Thus,

$$
a_n\sup_{s\in I\!R}|F(s)-\hat{F}(s)| = O((log\ n)^{\frac{2}{p+2}}n^{-\frac{(p-2)(p-2-2\delta)}{2p(p+2)}}) \qquad a.s. .
$$

Finally, we will check that the third term on the right-hand side of (27) is of the same order. For this goal, under (A) we may apply Corollary of Theorem 1 in Sadikova [12] and relation (21) in Newman [9] to find

$$
\begin{aligned}
&|Cov(1_{(-\infty,r]}(X_i), 1_{(-\infty,s]}(X_j))| \\
&\qquad \leq M_1|Cov^{1/3}(X_i,X_j)| \quad r,s \in I\!R, \qquad (28)
\end{aligned}
$$

where M_1 is defined in (A). Then, according to (28), we have

$$
\begin{aligned}
\sup_{r,s\in I\!R} &|\sum_{k=a_n+1}^{\infty}\varphi_k(r,s)| \\
&= \sup_{r,s\in I\!R}|\sum_{k=a_n+1}^{\infty} Cov(1_{(-\infty,r]}(X_1),1_{(-\infty,s]}(X_{k+1}))| \\
&\leq M_1\sum_{k=a_n+1}^{\infty}|Cov^{1/3}(X_1,X_{k+1})| \\
&= M_1 u(a_n) \leq C a_n^{-\frac{(p+3)(p-2)}{2(p+2)}}, \qquad (29)
\end{aligned}
$$

since condition (19) is satisfied for $\theta = \frac{(p+3)(p-2)}{2(p+2)} > 0$. Now, it is easy to check that

$$a_n^{-\frac{(p+3)(p-2)}{2(p+2)}} = n^{-\frac{(p-2)(p-2-2\delta)}{2p(p+2)}},$$

hence,

$$\sup_{r,s\in I\!R}|\sum_{k=a_n+1}^{\infty}\varphi_k(r,s)| = O(n^{-\frac{(p-2)(p-2-2\delta)}{2p(p+2)}}),$$

so the proof is concluded. □

The next theorem summarizes the previous result.

Theorem 4.1. *Suppose (A) holds. Under condition (19) for all $n \geq 1$, $\theta > 0$ and for every $0 < \gamma < 1/2$, we have*

$$\sup_{r,s\in I\!R}|\sum_{k=1}^{a_n}\hat{\varphi}_k(r,s) - \sum_{k=1}^{\infty}\varphi_k(r,s)| = O(n^{-\gamma}) \qquad a.s.\ , \qquad (30)$$

if $a_n = n^{\frac{p-2-2\delta}{p^2+3p}}$, with $\delta > 0$ and $p > 2$ chosen such that

$$\frac{(p-2)(p-2-2\delta)}{2p(p+2)} > \gamma.$$

Proof. Follow the arguments of the proof of Theorem 3.1, invoking Lemma 4.1 instead of Lemma 3.1. □

The next result implies the convergence rates for the $\hat{\Gamma}_n$.

Theorem 4.2. *Suppose (A) holds. Under condition (19) for all $n \geq 1$, $\theta > 0$ and for every $0 < \gamma < 1/2$, we have*

$$\sup_{r,s\in I\!R}|\hat{\Gamma}(r,s) - \Gamma(r,s)| = O(n^{-\gamma}) \qquad a.s.\ , \qquad (31)$$

if $a_n = n^{\frac{p-2-2\delta}{p^2+3p}}$, *with* $\delta > 0$ *and* $p > 2$ *chosen such that*

$$\frac{(p-2)(p-2-2\delta)}{2p(p+2)} > \gamma.$$

Proof. First write

$$\begin{aligned}
&\sup_{r,s\in\mathbb{R}} |\hat{\Gamma}(r,s) - \Gamma(r,s)| \\
&\quad \le \sup_{r,s\in\mathbb{R}} |\hat{F}_k(r,s) - F_k(r,s)| + 2\sup_{s\in\mathbb{R}} |\hat{F}(s) - F(s)| \\
&\qquad + \sup_{r,s\in\mathbb{R}} |\sum_{k=1}^{a_n} \hat{\varphi}_k(r,s) - \sum_{k=1}^{\infty} \varphi_k(r,s)| \\
&\qquad + \sup_{r,s\in\mathbb{R}} |\sum_{k=1}^{a_n} \hat{\varphi}_k(s,r) - \sum_{k=1}^{\infty} \varphi_k(s,r)|. \qquad (32)
\end{aligned}$$

Thus, the proof follows directly from Theorem 3.1 and 4.1. □

5 Concluding Remarks

Empirical process for independent data have been used for many years in statistics and probability theory. The need to model the dependence structure in data sets from many different subjects areas such as finance, insurance, telecommunications and reliability has led to new developments concerning the empirical process for dependent sequences. One such structure arises from negatively associated random variables. As mentioned in previous sections, we find almost sure convergence rates for the estimation of the two-dimensional distribution function of (X_1, X_{k+1}) without any condition on covariance structure of the variables. According these rates we estimate Γ, covariance function of the limit empirical process. For this purpose, we considered a convenient decrease rate of the covariances. The starting point for the derivation of these rates is a moment inequality. This inequality is used to estimate each of the terms that appear in the covariance function (1). Henriques and Oliveira [2] studies the properties of the histogram estimator for the distribution function of (X_1, X_{k+1}) under positive association. They proved almost sure and weak convergence of the estimator without any discussion about rates. However, we derive uniform strong convergence rates of the estimator for negatively associated samples without any restriction on the covariance structure of (X_1, X_{k+1}). Furthermore, we expand this convergence rates for the estimation of covariance operator of the limit empirical process.

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References

[1] Alam, K. and Saxena, K. M. L. (1981), Positive dependent in multivariate distribution. Commun. Statist-Theor. Meth., **A 10**, 1183–1196.

[2] Henriques, C. and Oliveira, P. E. (2003), Estimation of a two dimensional distribution function under association. J. Statist. Planning Inf., **113**, 137–150.

[3] Henriques, C. and Oliveira, P. E. (2005), Strong convergence rates for the estimation of a covariance operator for associated samples. Preprint, Pré-Publicações do Departamento de Matemática da Universidade de Coimbra 05-05.

[4] Joag-Dev, K. and Proschan, F. (1983), Negative association of random variables. with application. Ann. Statist., **11**, 286–295.

[5] Liang, H. Y. and Su, C. (1988), Convergence in the law of logarithm for NA sequences. Chinese Sci. Bull., **43**, 1919–1925 (in Chinese).

[6] Masry, E. (2002), Multivariate probability density estimation for associated processes: strong consistency and rates. Statist. Probab. Lett., **58**, 205–219.

[7] Matula, P. (1992), A note on the almost sure convergence of sums of negatively dependent random variables. Statist. Probab. Lett., **15**, 209–213.

[8] Matula, P. (1997), Probability and moment bounds for sums of negatively associated random variables. Theory Probab. Math. Stat., **55**, 135–141.

[9] Newman, C. M. (1980), Normal fluctuations and the FKG inequalities. Comm. Math. Phys., **74**, 119–128.

[10] Newman, C. (1984), Asymptotic independence and limit theorems for positively and negatively dependent random variables. Inequalities in Statistics and Probability, Y. L. Tong, ed, IMS Lecture Notes- Monograph Series, **5**, 127–140.

[11] Roussas, G. G. (1994), Asymptotic normality of random fields of positively or negatively associated processes. J. Multivariate Anal., **50**, 152–173.

[12] Sadikova, S. M. (1966), Two-dimensional analogies of an inequality of Esseen with applications to the central limit theorem. Theory Probab. Appl., **11**, 325–335.

[13] Shao, Q. M. (2000), A comparison theorem on maximal inequality between negatively associated and independent random variables. J. Theore. Probab., **13**, 343–356.

[14] Shao, Q. M. and Yu, H. (1996), Weak convergence for weighted empirical process of dependent sequences. Ann. Probab., **24**, 2098–2127.

[15] Su, C. and Chi, X. (1998), Some result on CLT for nonstationary NA sequences. Acta. Math. Appl. Sinica, **21**, 9–21.

[16] Su, C., Zhao, L. C., and Wang, Y. B. (1997), Moment inequalities and weak convergence for negatively associated sequences. Science in China, series A, **40**, 172–182.

[17] Yuan, M., Su, C., and Hu, T. (2003), A central limit theorem for random fields of negatively associated process. J. Theore. Probab., **16**, 309–323.