JIRSS (2006)
Vol. 5, Nos. 1-2, pp 1-8

A Test for Weibull IFR/DFR Alternatives Based on Type-2 with Replacement Censored Samples

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Abstract. This article presents a test based on quadratic form using Type-2 with replacement-censored sample for testing exponentiality against weibull IFR/DFR alternative. The percentile points and powers are simulated. The proposed test is compared with that of Bain and Engelhardt (1986) test. An example based on Type-2 censoring is also discussed.

1 Introduction

The weibull distribution is defined by the pdf

$$f(x;\theta,\beta) = \frac{\beta}{\theta} x^{\beta-1} e^{-x^{\beta}/\theta}, \quad x > 0, \ \theta, \beta > 0 \tag{1.1}$$

This distribution is quite popular as a life testing model and for many other applications where a skewed distribution is required. This model includes the exponential distribution with constant failure rate (CFR) for $\beta = 1$ and provide an increasing failure rate (IFR) for $\beta > 1$

Key words and phrases: Conditional distribution, nuisance parameter, percentile points, power of the test, shape parameter, weibull distribution.

and decreasing failure rate (DFR) for $\beta < 1$. Hence test for β is of interest.

Thoman et.al. (1969) have considered the problem of testing of hypothesis regarding the shape parameter based on complete samples. Bain and Engelhardt (1986) have proposed a modified version of Thoman et.al. (1969) test statistic whose asymptotic distribution is approximated to a chi-squared distribution. Lawless (1982), Bain and Engelhardt (1991) have discussed the problem of estimation and testing of shape parameter under Type-2 without replacement censoring and Type-1 censoring schemes. Very recentlty Muralidharan and Shanubhogue (2004) have obtained conditional test for Weibull DFR alternatives based on without replacement censored scheme. But there is less work done in the case of Type-2 with replacement censoring scheme because of complexity of finding distributions of the statistics obtained. In this article we propose a computationally simple test for testing $H_0 : \beta = 1$ against $H_1 : \beta > 1$ (or $H_1 : \beta < 1$) using Type-2 with replacement censored samples. The proposed test is also compared with that of Bain and Engelhardt (1986) test.

2 Derivation of the test

Many times observations of failures are naturally occurring in order. In this case, it is convenient to terminate the experiment after observing the first r failures from n units by replacing each failed item with a new item. In this section we derive a test statistic based on a Type-2 with replacement sample and study its properties.

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$, $r \leq n$, be a Type-2 with replacement-censored samples of a complete sample of size n from (1.1). Then the joint density is given by

$$f_{\underline{X}}(\underline{x};\theta,\beta) \propto f(x_{(1)})f(x_{(2)} - x_{(1)})\ldots f(x_{(r)} - x_{(r-1)})[F(x_{(r)})]^{n-1}.$$

Let $Y_i = X_i^{\beta},\ i = 1, 2, \ldots, n$ then Y_i follows an exponential distribution with density function

$$f_{Y_i}(y;\theta) = (1/\theta)e^{-y/\theta}, \quad y > 0, \theta > 0. \tag{2.1}$$

Let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(r)}$ be the corresponding type-2 with replacement-censored samples. Then

$$f_{\underline{Y}}(y;\theta) = (n/\theta)^r e^{-ny_{(r)}/\theta}, 0 < Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(r)} < \infty. \tag{2.2}$$

For known β, $T = Y_{(r)}$ is the complete sufficient statistic for θ. By making the transformation $Z_i = Y_{(i)} - Y_{(i-1)}$, $Y_{(0)} = 0$, $i = 1, 2, \ldots, r$, We get $\sum_{i=1}^{r} Z_i = Y_{(r)}$. Hence (2.2) reduces to

$$\begin{aligned} f_{\underline{Z}}(z;\theta) &= (n/\theta)^r e^{-n\sum_{i=1}^{r} z_i/\theta} \\ &= \prod_{i=1}^{r} \frac{n}{\theta} e^{-nz_i/\theta}. \end{aligned}$$

Therefore Z_i's are i.i.d exponential with parameter (θ/n) and hence $Y_{(r)}$ is gamma with parameter (θ/n) and r. Then the conditional pdf of $Y_{(1)}, Y_{(2)}, \ldots, Y_{(r-1)}$ given $T = t$ is obtained as

$$f(Y_{(1)}, Y_{(2)}, \ldots, Y_{(r-1)}|T = t) = \frac{\Gamma(r)}{t^{r-1}}, \quad 0 < Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(r-1)} < t \tag{2.3}$$

It is seen that this conditional density does not depend on the nuisance parameter θ. Hence we derive the test statistic for testing H_0 versus H_1 by treating the observations have come from (2.3). For this, we consider the quadratic form $Q = (\underline{Y} - \underline{\mu_0})'\Sigma_0^{-1}(\underline{Y} - \underline{\mu_0})$, where $\underline{Y}' = (Y_{(1)}, Y_{(2)}, \ldots, Y_{(r-1)})$, $\underline{\mu_0}' = (\mu_1, \mu_2, \ldots, \mu_{r-1})$; $\mu_i = E_{h_0}[Y_{(i)}|T = t]$ and $\Sigma_0 = ((\sigma_{ij}))$ is the conditional variance-covariance matrix of $\underline{Y}$ given $T = t$ computed under H_0. They can be obtained as follows:

We know that Z_i's are i.i.d exponential random variables with mean (θ/n) and $\sum_{i=1}^{r} Z_i = t$. Then the pdf of Z_i given $T = t$ is

$$f_{Z_i|T}(z_i|t) = \frac{(r-1)}{t}\Big(1 - \frac{z_i}{t}\Big)^{r-2}, \quad 0 < z_i < t \tag{2.4}$$

and

$$f_{Z_i,Z_j|T}(z_i, z_j|t) = \frac{(r-1)(r-2)}{t}\Big(1 - \frac{z_i}{t} - \frac{z_j}{t}\Big)^{r-3}, \quad 0 < z_i + z_j < t.$$

According to the theorem 1.6.7 of Reiss (1989), the distribution of the vector $(Z_1|T, Z_2|t, \ldots, Z_r|T)'$ is the same as that of $(V_1, V_2, \ldots, V_r)'$ where V's are the spacings of a random sample of size r from the uniform distribution on $(0, 1)$. Further, the author has given the asymptotic distribution of this vector of spacings and other related results (see corollary 1.6.10 of Reiss (1989)).

From (2.4) we obtain the moments of $Z_i|t$. Since $Y_{(i)} = \sum_{j=1}^{i} z_j$, under H_0

$$\mu_i = E_{H_0}(Y_{(i)}|t) = \sum_{j=1}^{i} E(Z_j|T = t) = \frac{it}{r} \tag{2.5}$$

$$
\begin{aligned}
\sigma_{ii} &= V_{H_0}(Y_{(i)}|t) \\
&= V_{H_0}(\sum_{j=1}^{i} Z_j|T=t) \\
&= \sum_{j=1}^{i} V_{H_0}(Z_j|T=t) - \sum_{j\neq k}^{i} V_{H_0}(Z_j|T=t)/(r-1) \\
&= \frac{i(r-i)v}{(r-1)}, \qquad (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{il} &= COV_{H_0}(Y_{(i)}, Y_{(l)}|t) \\
&= \sum_{j=1}^{i}\sum_{k=1}^{l} COV(Z_j, Z_k|T=t) \\
&= \sum_{j=1}^{i} V_{H_0}(Z_j|T=t) - \frac{1}{(r-1)}\sum_{j=1}^{i}\sum_{j\neq k=1}^{l} V_{H_0}(Z_j|T=t) \\
&= \frac{i(r-l)v}{(r-1)}, \qquad (2.7)
\end{aligned}
$$

$i = 1, 2, \ldots, r-1$; $l = 1, 2, \ldots, r-1$, $i < l$ and

$$
v = V_{H_0}(Z_j|T=t) = \frac{(r-i)t^2}{r^2(r+1)}.
$$

Using (2.6) and (2.7), the variance-covariance matrix Σ_0 of $(\underline{Y}|T=t)$ under H_0 is

$$
\Sigma_0 = \frac{v}{r-1}\begin{bmatrix} r-1 & r-2 & r-3 & \cdots & 1 \\ r-2 & 2(r-2) & 2(r-3) & \cdots & 2 \\ r-3 & 2(r-3) & 3(r-3) & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & r-1 \end{bmatrix}.
$$

Using Graybill (1969, pp.181), we get the inverse of Σ_0 as

$$
\Sigma_0^{-1} = \frac{r^2(r+1)}{t^2}\begin{bmatrix} 2/r & -1/r & 0 & \cdots & 0 \\ -1/r & 2/r & -1/r & \cdots & 0 \\ 0 & -1/r & 2/r & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2/r \end{bmatrix}
$$

Substituting the values of μ_i and Σ_0^{-1} in the expression for Q and on simplification, we obtain the test statistic as

$$Q = 2r(r+1)\sum_{i=1}^{r-1}\left(\frac{Y_{(i)}}{t}\right)\left[\left(\frac{Y_{(i)}}{t}\right) - \left(\frac{Y_{(i+1)}}{t}\right)\right] + (r^2 - 1).$$

Replacing t by its corresponding random variable T, we suggest the test statistics as

$$Q^* = 2r(r+1)\sum_{i=1}^{r-1} W_i(W_i - W_{i+1}) + (r^2 - 1), \tag{2.8}$$

where $W_i = (Y_{(i)}/Y_{(r)})$. The mean and variance of the test statistics under H_0 is given by

$$E(Q^*) = r - 1 \quad \text{and} \quad V(Q^*) = \frac{4r^2(r-1)}{(r+2)(r+3)}.$$

Since we could not find the expression for $E(Q^*)$ under H_1, we observed the direction of the test statistics by simulating its values for different n and β and under different censoring proportion $p(= r/n)$, $0 < p < 1$. They are presented in Table 1.

Table 1: $E(Q^*)$ for different values of β.

β / n	0.4	0.6	0.8	1.0	1.2	1.4	1.6	2.0
10	15.36	10.92	8.85	8.01	7.72	8.28	9.78	10.71
	13.30	9.66	7.59	7.02	6.89	7.19	8.39	9.04
20	34.18	22.77	18.35	17.03	17.09	18.47	23.71	27.35
	29.63	20.86	16.46	15.02	15.18	15.92	20.72	23.24
30	53.23	35.41	28.49	26.02	26.27	28.99	39.58	45.07
	47.58	31.49	25.33	23.06	23.45	25.34	33.40	39.37

(The first value corresponds to 10% censoring and second value corresponds to 20% censoring).

The table shows that $E_{H_1}(Q^*) > E_{H_0}(Q^*)$ for $0 < \beta < 1$ and for all n. For $\beta > 1$, $E_{H_1}(Q^*)$ decreases below $E_{H_0}(Q^*)$ for some range of β and then increases for $\beta > 1.4$. Thus the test procedure is to reject H_0 for large values of Q^* in the above ranges.

3 Simulation study

We obtain the upper tail percentile points of the distribution of Q by Monte Carlo method by generating 5000 random samples of different sizes n from weibull distribution with $\theta = \beta = 1$ and then construct type-2 with replacement censored samples under different censoring proportions. The results are given in Table 2.

Table 2: The upper percentile points of the distribution of Q^*.

n	$p = r/n = 0.10$		$p = r/n = 0.20$	
	0.95	0.99	0.95	0.99
10	13.9579	20.6123	17.0085	23.7507
20	27.8470	36.4925	30.5319	41.2601
30	38.5849	50.2322	41.8629	55.0654
50	60.1939	76.6189	66.3461	96.7664
70	80.5468	99.6361	88.9836	106.3851
90	99.8030	117.5192	111.6704	132.3699
120	129.7340	148.3228	145.6401	169.3696

We also compute the power of the test for different values of β and n. The proposed test is compared with the test proposed by Bain and Engelhardt (1986), say BE test and is given as follows: A size α test for testing $H_0 : \beta \leq \beta_0$ against $H_1 : \beta \geq \beta_0$, the test reject H_0 if $cr(\beta_0/\hat{\beta})^{1+\beta^2} < \chi^2_{\alpha(c(r-1))}$, where $c = 2/[(1+p^2)^2 p c_{22}]$; $p = r/n$, c_{22} is the asymptotic variance of $(\hat{\beta}/\beta)$ and $\hat{\beta}$ is the MLE of β. The values of c_{22} is well tabulated (see Bain and Engelhardt, 1986) for different values of n and p. Table 3 present the values of the powers of Q^* and BE test correspond to $\beta > 1$ for 5% level of significance under different censoring schemes.

Table 3: Power of the test for IFR alternatives

n	$\beta = 1.4$		$\beta = 1.6$		$\beta = 1.8$		$\beta = 2.0$	
	Q^*	BE	Q^*	BE	Q^*	BE	Q^*	BE
10	.063	.062	.108	.107	.149	.144	.185	.184
	.058	.055	.095	.101	.139	.137	.149	.144
20	.079	.073	.138	.133	.200	.198	.295	.290
	.076	.072	.115	.114	.179	.172	.272	.271
30	.111	.115	.206	.210	.339	.336	.482	.479
	.095	.111	.163	.155	.289	.286	.404	.402
50	.159	.165	.295	.288	.509	.502	.678	.677
	.136	.143	.256	.251	.460	.455	.617	.615
120	.250	.259	.561	.558	.815	.813	.963	.956
	.226	.230	516	.512	.779	.791	.949	.935

(The first value corresponds to 10% censoring and second value corresponds to 20% censoring).

From the above table it is seen that as the values of n and β increases the power of the proposed test fairs better than the BE test. Table 4 presents the powers correspond to $\beta < 1$ for Q^* test for both 1% and 5% level of significance.

Table 4: Power of the test for DFR alternatives

n	$\beta = 0.2$		$\beta = 0.4$		$\beta = 0.6$		$\beta = 0.8$	
	1%	5%	1%	5%	1%	5%	1%	5%
10	.492	.733	.155	.353	.067	.134	.016	.068
	.482	.728	.154	.341	.050	.126	.015	.062
20	.749	.913	.259	.497	.079	.194	.026	.076
	.735	.892	.241	.445	.069	.173	.021	.071
30	.909	.980	.363	.634	.099	.245	.028	.079
	.879	.971	.347	.599	.092	.234	.023	.075
50	.978	.999	.485	.789	.136	.311	.029	.079
	.974	.998	.470	.745	.126	.287	.027	.077
120	1.00	1.00	.899	.998	.238	.475	.039	.109
	1.00	1.00	.866	.962	.227	.463	.037	.108

(The first value corresponds to 10% censoring and second value corresponds to 20% censoring).

Thus from the above tables it is observed that the proposed test performs well for identifying both IFR and DFR alternatives under with replacement censored samples.

4 Example

We consider the data recorded based on a life test for a new insulating material. 25 specimens were tested simultaneously and the test was run until 15 of the specimens failed (for more details, see example 7.13 of Meeker and Escobar, 1998). Assuming that the data were recorded under with replacement scheme, we obtain the percentile points correspond to $n = 25$ and $r = 15$ as 62.83 and 88.14 respectively for upper 5% and 1% percentile points. Under the null hypothesis, the computed value of Q^* is 112.78, which is larger than the percentile points corresponds to 1% and hence the test is rejected. The p-value correspond to this test is 0.004. The power corresponds to the specific alternatives say, $H_1 : \beta = 1.8$ correspond to 1% and 5% are 0.11 and 0.18 respectively. Similarly, for $H_1 : \beta = 0.5$, the powers are obtained as 0.24 and 0.35 respectively for 1% and 5% cutt-off points.

Acknowledgements

The author thanks the Editor for their comments. The author also acknowledges the University Grants Commission, New Delhi for sanctioning the research project no. F:6-7/2001 (SR-1).

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