

## A Note on the Strong Law of Large Numbers

V. Fakoor, H. A. Azarnoosh

Department of Statistics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Iran. (fakoor@math.um.ac.ir, azarnush@math.um.ac.ir)

**Abstract.** Petrov (1996) proved the connection between general moment conditions and the applicability of the strong law of large numbers to a sequence of pairwise independent and identically distributed random variables. This note examines this connection to a sequence of pairwise *negative quadrant dependent* (NQD) and identically distributed random variables. As a consequence of the main theorem (Theorem 2.1), we arrive at an improvement of Marcinkiewicz–Zygmund theorem for pairwise NQD random variables.

### 1 Introduction

Let  $\{X_n, n \geq 1\}$  be a sequence of independent and identically distributed random variables. There are two famous theorems on the strong law of large numbers for such a sequence: Kolmogorov's theorem and Marcinkiewicz-Zygmund theorem (see a.g. Loeve, 1963 or Stout, 1974). In what follows, we put  $S_n = \sum_{k=1}^n X_k$ . According to Kolmogorov's theorem, there exists a constant  $b$  such that

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$S_n/n \rightarrow b$  a.s. if and only if  $E|X_1| < \infty$ ; if the later condition is satisfied then  $b = EX_1$ . By the Marcinkiewicz-Zygmund theorem, if  $0 < p < 2$  then the relations  $(S_n - nb)/n^{1/p} \rightarrow 0$  a.s., and  $E|X_1|^p < \infty$  are equivalent. Here  $b = 0$  if  $0 < p < 1$  and  $b = EX_1$  if  $1 \leq p < 2$ .

Etemadi (1981) proved that Kolmogorov's theorem remains true if we replace the independence condition by the weaker condition of pairwise independence of random variables  $X_1, X_2, \dots$ . Petrov (1996) proved the connection between general moment conditions and the applicability of the strong law of large numbers to a sequence of pairwise independent and identically distributed random variables. We extend his result to pairwise NQD random variables.

**Definition 1.1.** A sequence of random variables  $\{X_n, n \geq 1\}$  is said to be pairwise negative quadrant dependent (NQD) if

$$P(X_i > x, X_j > y) \leq P(X_i > x)P(X_j > y)$$

for all  $x, y \in \mathbb{R}$  and for all  $i, j \geq 1, i \neq j$ .

This definition was introduced by Lehmann (1966). Matula (1992) proved that Kolmogorov's theorem remains true if we replace the pairwise independence condition by the weaker condition of pairwise NQD of random variables  $X_1, X_2, \dots$

## 2 Main result

In order to prove the main theorem (Theorem 2.1), we shall state the following lemmas for later references. These lemmas can be found in Matula (1992).

**Lemma 2.1.** *Let  $(\Omega, F, P)$  be a probability space and  $\{A_n, n \in N\}$  a sequence of events. If  $\sum_{n=1}^{\infty} P(A_n) < \infty$ , then  $P(\limsup A_n) = 0$ , if  $\sum_{n=1}^{\infty} P(A_n) = \infty$  and  $P(A_k \cap A_m) \leq P(A_k)P(A_m)$  for  $k \neq m$ , then  $P(\limsup A_n) = 1$ .*

**Lemma 2.2.** *If  $\{X_n, n \in N\}$  is a sequence of pairwise NQD random variables,  $\{f_n, n \in N\}$  a sequence of nondecreasing functions*

$f_n : \mathbb{R} \rightarrow \mathbb{R}$ , then  $\{f_n(X_n), n \in N\}$  are also pairwise NQD.

**Theorem 2.1.** Let  $\{X_n, n \geq 1\}$  be a sequence of pairwise NQD random variables. Suppose

$$a_n = f^{-1}(n) \tag{1}$$

for an even continuous, positive and strictly increasing function  $f$  on  $\mathbb{R}^+$  such that  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If

$$\frac{S_n}{a_n} \rightarrow 0 \quad a.s. \tag{2}$$

then

$$Ef(X_1) < \infty. \tag{3}$$

**Proof.** By the properties of  $f$ , since  $a_{n-1}/a_n \leq 1$ , we observe that the equality

$$\frac{X_n}{a_n} = \frac{S_n}{a_n} - \frac{S_{n-1}}{a_{n-1}} \cdot \frac{a_{n-1}}{a_n}$$

and (2), imply the relation

$$\frac{X_n}{a_n} \rightarrow 0 \quad a.s.$$

So,

$$\frac{X_n^+}{a_n} \rightarrow 0 \quad a.s. \quad , \quad \frac{X_n^-}{a_n} \rightarrow 0 \quad a.s.,$$

where  $x^+ = \max(0, x)$  and  $x^- = \max(0, -x)$ .

It is easy to verify that  $\{-X_n, n \geq 1\}$  is a sequence of pairwise NQD random variables, thus taking into account Lemma 2.2 we see that  $\{X_n^+, n \geq 1\}$  and  $\{X_n^-, n \geq 1\}$  are pairwise NQD. Defining the following events ,

$$A_n = [X_n^+ > \frac{a_n}{3}], \quad B_n = [X_n^- > \frac{a_n}{3}],$$

for  $n \geq 1$ , we have

$$P(A_k \cap A_l) \leq P(A_k)P(A_l), \quad P(B_k \cap B_l) \leq P(B_k)P(B_l) \quad \text{for } k \neq l.$$

By lemma 2.1 if  $\sum_{n=1}^{\infty} P(A_n)$  diverges then  $P(A_n \text{ i.o. }) = 1$  contrary

to the almost sure convergence of  $X_n^+/a_n$  to zero. Therefore

$$\sum_{n=1}^{\infty} P(X_n^+ > \frac{a_n}{3}) < \infty.$$

The same argument for  $X_n^-/a_n$  yields

$$\sum_{n=1}^{\infty} P(X_n^- > \frac{a_n}{3}) < \infty.$$

Thus

$$\sum_{n=1}^{\infty} P(|X_n| \geq a_n) \leq \sum_{n=1}^{\infty} P(X_n^+ > \frac{a_n}{3}) + \sum_{n=1}^{\infty} P(X_n^- > \frac{a_n}{3}) < \infty,$$

or

$$\sum_{n=1}^{\infty} P(|X_1| \geq f^{-1}(n)) < \infty.$$

Consequently,

$$\sum_{n=1}^{\infty} P(f(X_1) \geq n) < \infty. \tag{4}$$

For an arbitrary random variable  $Y$  the conditions  $\sum_{n=1}^{\infty} P(|Y| \geq n) < \infty$  and  $E|Y| < \infty$  are equivalent. (See e.g. Chung (2001), page 45.) Therefore, it follows from (4) that (3) holds. Theorem 2.1 is proved. ■

If we consider  $f(x) = |x|^p$ ,  $p > 0$ , we have  $a_n = f^{-1}(n) = n^{1/p}$ . By Theorem 2.1, the relation

$$\frac{S_n - nb}{n^{1/p}} \rightarrow 0 \quad a.s.,$$

for some  $b \in \mathbb{R}$  implies that  $E|X_1|^p < \infty$  assuming pairwise NQD. Thus we arrive at an improvement of Marcinkiewicz-Zygmund theorem for pairwise NQD random variables.

The following theorem can be found in Petrov (1996). Note that there is no independence (or dependence) condition in the following theorem.

**Theorem 2.2.** *Let  $\{X_n, n \geq 1\}$  be a sequence of identically distributed random variables. If*

$$\sum_{k=n}^{\infty} \frac{1}{a_k} = O\left(\frac{n}{a_n}\right), \tag{5}$$

then (3) implies (2). ■

It follows from Theorems 2.1 and 2.2 that if  $\{X_n, n \geq 1\}$  is a sequence of pairwise NQD identically distributed random variables and if (5) is satisfied then (2) and (3) are equivalent. Note that we cannot omit condition (5) in Theorem 2.2. For an example see Petrov (1996).

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## References

- Chung, K. L. (2001), A Course in Probability Theory. New York: Academic Press.
- Etemadi, N. (1981), An elementary proof of the strong law of large numbers, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **55**, 119–122.
- Lehmann, E. (1966), Some concepts of dependence. *Ann. Math. Statist.* **37**, 1137–1153.
- Loeve, M. (1963), Probability Theory. Van Nostrand, Princeton, 3rd ed.
- Matula, P. (1992), A note on the almost sure convergence of negatively dependent variables. *Statist. Probab. Lett.*, **15**, 209–213.
- Petrov, V. V. (1996), On the Strong law of large numbers. *Statist. Probab. Lett.*, **26**, 377–380.
- Stout, W. F. (1974), Almost Sure Convergence. New York: Academic Press.

