Record Range of Uniform Distribution

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Abstract. We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $R_{nl}$ be the largest observation after observing $n$th record and $R_{(ns)}$ be the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range. We will consider some distributional properties of $W_{nr}$ when $f(x) = 1$, $0 \leq x \leq 1$.

1 Introduction

Let $\{X_i, i = 1, 2, ..\}$ be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebegue measure) distribution function $F(x)$ with pdf $f(x)$. Let $R_{U(1)} = X_1$, $R_{U(2)}$, ..., be the upper records and $R_{L(1)}, R_{L(2)}, ...$ be lower records of $\{X_i, i = 1, 2, ..\}$. For various properties of record values see Ahsanullah (2005) Arnold et. al.(1998).

Suppose $R_{nl}$ is the largest observation after observing $n$th record and $R_{(ns)}$ is the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range. The

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The joint pdf of \( f_{(nl, (ns))} \) of \( R_{nl} \) and \( R_{(ns)} \) is given by (see Arnold et al. 1998, p. 275) as
\[
f_{(nl, (ns))}(x, y) = \frac{2^{n-1}}{(n-2)!} \left[-\ln(F(y) + F(x))\right]^{n-2} f(x) f(y), \quad (1.1)
\]
\(-\infty < x < y < \infty\)

The pdf of \( f_{W_{nr}} \) of \( W_{nr} \) is given by
\[
f_{W_{nr}}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} \left[-\ln(F(w+u)+F(u))\right]^{n-2} f(w+u) f(u) \, du \quad (1.2)
\]
Suppose \( X_i' \)'s are distributed as uniform with
\[
f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)
\]
Using (1.3) in (1.2), we obtain
\[
f_{W_{nr}}(w) = \begin{cases} \frac{2^{n-1}}{\Gamma(n-1)} [-\ln(1-w)]^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases} \quad (1.4)
\]
Figure 1.1 gives the pdf of \( W_{nr} \) for \( n = 10 \) when \( X_i' \)'s are distributed as uniform.

In this paper we will consider distributional properties of \( W_{nr} \) for the case \( X_i' \)'s are distributed as uniform distribution.

2 Main Results

Lemma 2.1. For \( n \geq 2 \) and \( 0 < x < 1 \),
\[
F_{W_{nr}}(x) = \Gamma_{-2\ln(1-x)}(n-1),
\]
where
\[
\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} \, du
\]
Proof.
\[
F_{W_{nr}}(x) = \int_0^x \frac{2^{n-1}}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} du
= \int_0^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} \, dt
= \Gamma_{-2\ln(1-x)}(n-1). \quad \square
\]
Figure 1.1: \( f(x) = \frac{2^9(1-x)}{\Gamma(9)} \left(-\ln(1-x)\right)^8 \)

Remark 2.1.

\[ 1 - F_{W_{nr}}(x) = (1 - x)^2 \sum_{j=0}^{n-2} \frac{(-2 \ln(1-x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2 \]

Lemma 2.2.

\[ F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1-x)} \]

Proof.

\[
F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \Gamma_{-2 \ln(1-x)}(n-1) - \Gamma_{-2 \ln(1-x)}(n) \\
= \frac{1}{\Gamma(n)}(-2 \ln(1-x))^{n-1}e^{-2 \ln(1-x)} \\
= \frac{(1-x)^2}{\Gamma(n)}(-2 \ln(1-x))^{n-1} \\
= f_{W_{n+1r}}(x) \frac{1}{2(1-x)}. \]
\[\mu_{nr}^p = E(W_{nr})^p = \frac{2^{n-1}}{(n-2)!} \int_0^1 w^p (1 - w) [-\ln(1 - u)]^{n-2} dw\]
\[= \frac{2^{n-1}(n-2)!}{(n-2)!} \int_0^1 (1 - w)^p w [-\ln(w)]^{n-2} dw\]
\[= 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(2 + k)^{n-1}} \tag{2.1}\]

Using \(p = 1\) and \(p = 2\), we can get the mean and variance of \(W_{nr}\) as

\[E(W_{nr}) = 1 - \left(\frac{2}{3}\right)^{n-1}\]

and

\[Var(W_{nr}) = \left(\frac{1}{2}\right)^{n-1} - \left(\frac{4}{9}\right)^{n-1}. \□\]

**Theorem 2.1.** Let \(\mu_n^r = E(W_{nr}^r)\), then for \(n \geq 2\) and \(r = 1, 2, \ldots\)

\[(r + 2)\mu_n^r - r\mu_{n-1}^r = 2\mu_{n-1}^r. \tag{2.2}\]

**Proof.**

\[r(\mu_n^{r-1} - \mu_n^r)\]
\[= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 [(1 - w)^2 rw^{r-1} [-\ln(1 - w)]^{n-2} dw\]
\[= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r [2(1 - w)[-\ln(1 - w)]^{n-2} dw\]
\[- \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1 - w)^2 w^{r-1} [-\ln(1 - w)]^{n-3} \frac{1}{1 - w} dw\]
\[= 2\mu_n^r - 2\mu_{n-1}^r.\]

On simplification we get the result. \(\Box\)

**Theorem 2.2.** For \(n \geq 2\), \(p > 0\),

\[\mu_{nr}^p - \mu_{nr}^{p+1} = 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3 + k)^{n-1}}\]
Proof.

\[ \mu^p_{nr} - \mu^{p+1}_{nr} = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1 - w) [-\ln(1 - w)]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p(1 - w)^2 [-\ln(1 - w)]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1 - w)^p w^2 [-\ln(w)]^{n-2} dw \]

\[ = 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3 + k)^{n-1}} \quad (2.3) \]

Theorem 2.3. For \( n \geq 1 \),

\[ 1 - W_{n+1} \overset{d}{=} \Pi_{j=1}^n V_j, \quad (2.4) \]

where \( V_1, V_2, ..., V_{n-1} \) are i.i.d. with \( F(v) = v^2, \ 0 < v < 1 \).

Proof. We will show first that

\[ 1 - W_{n+1} \overset{d}{=} (1 - W_{nr}) V_n, \]

where \( V_n, \) is independent of \( 1 - W_{nr} \) and is distributed with pdf as \( f_V(v) = 2v, \ 0 < v < 1 \).

Let \( Y_{n-1} = (1 - W_{nr}) V_n, \ n \geq 2, \) and \( f_n \) be the pdf of \( 1 - W_{nr}, \) then

\[ f_n = \frac{2^{n-1}w}{\Gamma(n-1)}[-\ln(1 - w)]^{n-2}, \ 0 < w < 1, \]

\[ F(y) = P(Y_{n+1} \leq y) = P(1 - W_{nr})V_n \leq y) \]

\[ = y^2 + \int_y^1 F_n(t) \frac{2v}{v^2} dv, \ \text{where} \ F_{n-1} \ is \ the \ df \ of \ Y_n \]

\[ = y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^2} dt \]

\[ = y^2 + y^2 \int_y^1 F_n(t) \frac{1}{t^2} dt \]

\[ = y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \quad (2.5) \]
Differentiating both sides of (2.5), with respect to $y$, we obtain

$$f(y) = f_n(y) - f_n(y) + 2y \int_y^1 f_n(t) \frac{1}{t^2} dt$$

i.e.

$$\frac{f(y)}{y} = 2 \int_y^1 f_n(t) \frac{1}{t^2} dt$$

$$= 2 \int_y^1 \frac{2^{n-1} t}{\Gamma(n-1)} [-\ln(1-t)]^{n-2} \frac{1}{t^2} dt$$

$$= \frac{2^n}{\Gamma(n-1)} [-\ln t]^{n-1} - \frac{1}{n-1} y$$

$$= \frac{2^n}{\Gamma(n)} [-\ln y]^{n-1}$$

Hence

$$f(y) = \frac{2^n}{\Gamma(n-1)} y [-\ln y]^{n-1}$$

which is the pdf of $Y = 1 - W_{n+1r}$.

Note that the sequence $Y_2, Y_3, ...$ forms a Markov chain. □

Using (2.6), we have the following representation of $W_{nr}$ for

$$1 - W_{n+1r} \overset{d}{=} \Pi_{j=1}^n V_j, \quad n \geq 1$$

(2.8)

where $V_1, V_2, ..., V_{n-1}$ are i.i.d. with $F(v) = v^2, \quad 0 < v < 1.$

The conditional expectation of

$$1 - W_{nr}|1 - W_{mr} = x, \quad 2 \leq m < n - 1, \quad 0 < x < 1,$$

is

$$E(1 - W_{nr}|1 - W_{mr} = x) = x \left(\frac{2}{3}\right)^{n-m}.$$

Thus

$$\text{Cov}(W_{nr}W_{mr}) = \left(\frac{2}{3}\right)^{n-m} \text{Var}(W_{mr})$$

$$= \left(\frac{2}{3}\right)^{n-m} \left[\left(\frac{1}{2}\right)^{m-1} - \left(\frac{4}{9}\right)^{m-1}\right].$$
The correlation coefficient $\rho_{m,n}$ between $W_{nr}$ and $W_{mr}$ is given by

$$
\rho_{m,n} = \frac{\left(\frac{3}{2}\right)^{n-m}\sqrt{\left[\left(\frac{1}{2}\right)^{m-1} - \left(\frac{1}{2}\right)^{n-1}\right]}}{\left[\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{n-1}\right]} = \sqrt{\left[\left(\frac{3}{2}\right)^{m-1} - 1\right]}
$$

for any fixed $m$ as $n \to \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \leq m \leq n = 5$.

<table>
<thead>
<tr>
<th>m</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>2</td>
<td>$\frac{1}{18}$</td>
<td>$\frac{1}{27}$</td>
<td>$\frac{2}{81}$</td>
<td>$\frac{4}{243}$</td>
</tr>
<tr>
<td>n</td>
<td>3</td>
<td>$\frac{1}{27}$</td>
<td>$\frac{17}{324}$</td>
<td>$\frac{17}{486}$</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>$\frac{2}{81}$</td>
<td>$\frac{17}{486}$</td>
<td>$\frac{3862}{8748}$</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>$\frac{4}{243}$</td>
<td>$\frac{17}{729}$</td>
<td>$\frac{217}{8748}$</td>
</tr>
</tbody>
</table>

**Theorem 2.4.** The joint pdf $f_{m,n}^*$ of $W_{mr}$ and $W_{nr}$, $2 \leq m \leq n$, is given by

$$
f_{m,n}^*(x, y) = \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{n-1} (-\ln(1-x))^{m-2} \times (-\ln(1-x) + \ln(1-y))^{n-m-1} \frac{1-y}{(1-x)},
$$

$0 < x < y < 1$.

**Proof.** Let $U_1 = \Pi_{j=1}^m V_j$ and $U_2 = \Pi_{j=1}^n V_{m+j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$
f_{U_1U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m-1)} (-2 \ln u_1)^{m-2} \frac{2u_2}{\Gamma(n-m)} (-2 \ln u_2)^{n-m-1}.
$$

Let $T_1 = U_1$ and $T_2 = U_1 U_2$, then the joint pdf of $T_1$ and $T_2$ is

$$
f_{T_1T_2}(t_1, t_2) = \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)} (-2 \ln t_1)^{m-2} \frac{t_2}{t_1} (-2 \ln \frac{t_2}{t_1})^{n-m-1}
$$

$$
= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)} (-\ln t_1)^{m-2} \frac{t_2}{t_1} (-\ln t_2 - \ln t_1)^{n-m-1}
$$
Sustituting $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, We obtain the joint pdf of $W_{mr}$ and $W_{nr}$, $2 \leq m < n$ as

$$f_{m,n}^*(x, y) = \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln(1 - x))^{m-2} \times \left[-\ln(1 - x) + \ln(1 - y)\right]^{n-m-1} \frac{1 - y}{1 - x},$$

$$0 < x < y < 1. \quad \Box$$

**Theorem 2.5.** For $m \geq 2$, $p \geq 0$ and $q \geq 0$,

$$E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q + 3} E(W_{mr})^{p+q+1}$$

**Proof.**

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] = \int_0^1 \int_x^1 [(x)^p(y)^q(1 - y)] f_{m,m+1}^*(x, y)dydx$$

$$= \int_0^1 [(x)^p \frac{1}{\Gamma(m-1)} 2^{n-1}(-\ln(1 - x))^{m-2} \frac{1}{1 - x} H(x)]dx, (2.9)$$

where

$$H(x) = \int_x^1 (y)^q(1 - y)(1 - y)dy$$

$$= \int_x^1 (y)^q(1 - y)^2dy$$

$$= \frac{y^{q+1}}{q+1} (1 - y)^2 \bigg|_x^1 + \int_x^1 \frac{2y^{q+1}}{\theta^{q+1}(q+1)}(1 - y)dy$$

$$= -\frac{x^{q+1}}{q+1} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)}(1 - y)dy$$

Substituting in (2.9), we obtain

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] = \int_0^1 [(x)^p(-\ln(1 - x))^{m-2} \frac{1}{1 - x} H(x)]dx$$

$$= -\frac{2x^{q+1}}{q+1} E((W_{mr})^{p+q+1}) + 2 \int_x^1 \frac{y^{q+1}}{(q+1)}(1 - y)dy \frac{d}{dx}$$

$$= -\frac{2}{q + 1} E((W_{mr})^{p+q+1}) + \frac{2}{q + 1} E(W_{mr})^p(W_{m+1r})^{q+1}$$
Thus
\[ E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q+1}{q+3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q+3} E(W_{mr})^{p+q+1}. \]

Theorem 2.6. For \( m \geq 2, n > m \geq 2, p \geq 0 \) and \( q \geq 0 \),
\[ E(W_{mr})^p(W_{nr})^{q+1} = \frac{q+1}{q+3} E[(W_{mr})^p(W_{nr})^q] + \frac{2}{q+3} E((W_{mr})^p(W_{n-1r})^q) \]

Proof.
\[ E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}] \]
\[ = \int_0^1 \int_x^1 [(x)^p((y)^q(1-y))] f^*_{m,n}(x,y) dy dx \]
\[ = \int_0^1 [(x)^p \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{m+k-1} \]
\[ \cdot (-\ln(1-x))^{m-2} \frac{1}{(1-x)} H(x) dx, \quad (2.10) \]

where
\[ \frac{1}{\Gamma(n-m)} H(x) \]
\[ = \int_x^1 ((y)^q(1-y))[-\ln(1-x) + \ln(1-y)]^{n-m-1} \]
\[ \times (1-y) dy \]
\[ = \int_x^1 \frac{y^q}{q}[-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2 dy \]
\[ = \frac{y^{q+1}}{(q+1)}[-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2 |_{x}^{1} \]
\[ - \int_x^1 \frac{y^{q+1}}{q+1} d[-\ln(1-x) + \ln(1-y)]^{n-m-1} \]
\[ \times (1-y)^2 dy \]
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\[\begin{align*}
&= -\int_x^1 \frac{y^{q+1}}{q+1} \frac{d}{dy}[-\ln(1-x) + \ln(1-y)]^{n-m-1} \\
&\times (1-y)^2 \, dy \\
&= 2 \int_x^1 \frac{y^{q+1}}{q+1} \left[-\ln(1-x) + \ln(1-y)\right]^{n-m-1}(1-y) \\
&+ \int_x^1 \frac{y^{q+1}}{\theta^q(q+1)} (n-m-1) \left[-\ln(1-x) + \ln(1-y)\right]^{n-m-2} \\
&\times (1-y^-\theta) \, dy \\
\end{align*}\]

Substituting \(H(x)\) in (2.10), we obtain

\[E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{nr})^{q+1}] = 2^{q+1} E((W_{mr})^p(W_{nr})^{q+1}) - \frac{1}{q+1} E((W_{mr})^p(W_{n-1r})^{q+1})\]

On simplification, we obtain

\[E((W_{mr})^p(W_{nr})^{q+1}) = \frac{q+1}{q+3} E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3} E((W_{mr})^p(W_{n-1r})^{q+1})\]

\[\square\]

\textbf{Entropy of} \(W_{nr}\).

The entropy of \(W_{nr}\) is given in the following theorem.

\textbf{Theorem 2.7}. The entropy, \(I_n\) of \(W_{nr}\), \(n \geq 2\), is given by

\[I_n = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2,\]

where \(\Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)}\).

\textbf{Proof}.

\[I_n = E(-\ln f_{W_{nr}}) = \int_0^1 [\ln \Gamma(n-1) - (n-1) \ln 2 - \ln(1-u) - (n-2) \ln(-\ln(1-u))] \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} \, du = [\ln \Gamma(n-1) - (n-1) \ln 2 - H_1 - H_2], \quad (2.11)\]
where

\[ H_1 = \int_0^1 \ln(1-u) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} = -\frac{n-1}{2}, \]

\[ H_2 = \int_0^1 (n-2)\ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} du \]

Substituting \(-\ln(1-u) = t\), we obtain

\[ H_2 = \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-2t} dt \]

\[ = \frac{n-2}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-t} dt - (n-2) \ln 2 \]

\[ = (n-2)[\Psi(n-1) - \ln 2]. \]

Substituting \(H_1\) and \(H_2\) in (2.11), we obtain

\[ I_n = \ln \Gamma(n-1) - (n-1) \ln 2 + \frac{n-1}{2} \]

\[ - (n-2)[\Psi(n-1) - \ln 2]. \]

\[ = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2. \quad \square (2.12) \]

The following table gives \(-I_n\) for \(n = 3\) to 10.

<table>
<thead>
<tr>
<th>(n)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-I_n)</td>
<td>0.1159</td>
<td>0.3456</td>
<td>0.6698</td>
<td>1.0396</td>
<td>1.4363</td>
<td>1.8506</td>
<td>2.2775</td>
<td>2.7141</td>
</tr>
</tbody>
</table>

We will consider the estimation \(\theta\) based on the record range. when \(X_1, X_2, \ldots\) are i.i.d with \(f(x) = \frac{1}{\theta}, \quad 0 < x < \theta\).

**Theorem 2.8.** The minimum variance linear unbiased estimator of \(\theta\) is

\[ \hat{\theta} = \frac{1}{3(2^n-1)} [3.2^{n-1} W_{n+1r} - 2^{n-2} W_{nr} - 2^{n-3} W_{n-1r} - \ldots - W_{2r}] \]

and

\[ \text{Var}(\hat{\theta}) = \frac{\theta^2}{2(2^n-1)}. \]
Proof. Let

\[ Z_1 = d_1 W_{2r}, \quad d_1 = 3.2^{\frac{1}{2}} \]
\[ Z_2 = d_2 (W_{3r} - \frac{2}{3} W_{2r}), \quad d_2 = 3.2 \]
\[ \vdots \]
\[ Z_n = d_n (W_{n+1r} - \frac{2}{3} W_{nr}), \quad d_n = 3.2^{\frac{n}{2}} \]
\[ Z' = (Z_1, Z_2, ..., Z_n), \]

then \( E(Z') = A\theta \), where

\[ A' = (2^{\frac{1}{2}}, 2, ..., 2^{\frac{n}{2}}). A'A = 2(2^n - 1). \]

Then the minimum variance linear unbiased estimator (MVLUE) \( \hat{\theta} \) of \( \theta \) (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is

\[ \hat{\theta} = (A'A)^{-1} A'Z \]
\[ = \frac{1}{2(2^n - 1)} \left[ 2^{\frac{1}{2}} Z_1 + 2 Z_2 + ... + 2^{n-2} Z_n \right] \]
\[ = \frac{1}{2(2^n - 1)} \left[ 3.2^n W_{n+1r} - 2^{n-1} W_{nr} - 2^{n-2} W_{n-1r} - 2 W_{2r} \right]. \quad (2.13) \]

\[ Var(\hat{\theta}) = \theta^2 (A'A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square \]

For example, if \( n = 4 \), then

\[ \hat{\theta} = \frac{1}{15} [24W_{5r} - 4W_{4r} - 2W_{3r} - W_{2r}] \]

and

\[ Var(\hat{\theta}) = \frac{\theta^2}{30}. \]

Table 2.3. Coefficient of \( W_{nr} \) in MVLUE of \( \theta \).
Let \( \tilde{\theta} = c\hat{\theta} \), then bias of \( \tilde{\theta} \) is \((c - 1)\theta\) and mean squared error (MSE) of \( \tilde{\theta} \) is \( \text{MSE}(\tilde{\theta}) = c^2 \frac{\theta^2}{2^n - 1} + (c - 1)^2 \theta^2 \). The MSE of \( \tilde{\theta} \) will be minimum if \( c = \frac{2n + 1 - 2}{2^n - 1} \).

The bias of \( \tilde{\theta} = (c - 1)\theta = \frac{1}{2^n - 1} \) and MSE(\( \tilde{\theta} \)) = \( \frac{1}{2^n - 1} \).

**Prediction of** \( W_{n+sr} \).

We consider the prediction of \( W_{n+sr} \) based on \( W_{2r}, W_{3r}, \ldots, W_{nr} \).

**Theorem 2.5.** The best linear least squares predictor, \( W^*_{n+sr} \) of \( W_{n+sr} \) based on \( W_{2r}, W_{3r}, \ldots, W_{nr} \) is \( \theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr} \).

**Proof.** The best linear least squares predictor, \( W^*_{n+sr} \) of \( W_{n+sr} \) based on \( W_{2r}, W_{3r}, \ldots, W_{nr} \) is

\[
W^*_{n+sr} = E(W_{n+sr}|W_{2r} = x_2, W_{3r} = x_3, \ldots, W_{nr} = x_n) \\
= E(W_{n+sr}|W_{nr} = x_n), \text{ by Markov property of } W_{2r}, W_{3r}, \ldots \\
= \theta[1 - (\frac{2}{3})^s] + x_n(\frac{2}{3})^s .
\]

**References**

