Record Range of Uniform Distribution

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Abstract. We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $R_{nl}$ be the largest observation after observing $n$th record and $R_{(ns)}$ be the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range. We will consider some distributional properties of $W_{nr}$ when $f(x) = 1$, $0 \leq x \leq 1$.

1 Introduction

Let $\{X_i, i = 1, 2, ..\}$ be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebegue measure ) distribution function $F(x)$ with pdf $f(x)$. Let $R_{U(1)} = X_1, R_{U(2)}, ...$, be the upper records and $R_{L(1)}, R_{L(2)}, ...$, be lower records of $\{X_i, i = 1, 2, ..\}$. For various properties of record values see Ahsanullah( 2005) Arnold et. al.(1998).

Suppose $R_{nl}$ is the largest observation after observing $n$th record and $R_{(ns)}$ is the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range.

Key words and phrases: Minimum variance linear unbiased estimation, moments, record range, uniform distribution.
joint pdf of \( f_{(nl,(ns))} \) of \( R_{nl} \) and \( R_{(ns)} \) is given by (see Arnold et al. 1998, p. 275) as

\[
f_{(nl,(ns))}(x, y) = \frac{2^{n-1}}{(n-2)!} [-\ln(F(y) + F(x))]^{n-2} f(x) f(y), \quad (1.1)
\]

\(-\infty < x < y < \infty\)

The pdf of \( f_{W_{nr}} \) of \( W_{nr} \) is given by

\[
f_{W_{nr}}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} [-\ln(F(w+u)+F(u))]^{n-2} f(w+u) f(u) \, du \quad (1.2)
\]

Suppose \( X_i' \)'s are distributed as uniform with

\[
f(x) = \begin{cases} 
1, & 0 < x < 1 \\
0, & \text{otherwise}
\end{cases} \quad (1.3)
\]

Using (1.3) in (1.2), we obtain

\[
f_{W_{nr}}(w) = \begin{cases} 
\frac{2^{n-1}(1-w)}{\Gamma(n-1)} [-\ln(1-w)]^{n-2}, & 0 < w < 1, n \geq 2 \\
0, & \text{otherwise}
\end{cases} \quad (1.4)
\]

Figure 1.1 gives the pdf of \( W_{nr} \) for \( n = 10 \) when \( X_i' \)'s are distributed as uniform.

In this paper we will consider distributional properties of \( W_{nr} \) for the case \( X_i' \)'s are distributed as uniform distribution.

### 2 Main Results

**Lemma 2.1.** For \( n \geq 2 \) and \( 0 < x < 1 \),

\[
F_{W_{nr}}(x) = \Gamma_{-2\ln(1-x)}(n-1),
\]

where

\[
\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du
\]

**Proof.**

\[
F_{W_{nr}}(x) = \int_0^x \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} du
\]

\[
= \int_0^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} dt
\]

\[
= \Gamma_{-2\ln(1-x)}(n-1). \quad \square
\]
Figure 1.1: $f(x) = \frac{2^9(1-x)}{\Gamma(9)}(-\ln(1-x))^8$

Remark 2.1.

$$1 - F_{W_{n,r}}(x) = (1 - x)^2 \sum_{j=0}^{n-2} \frac{(-2\ln(1-x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2$$

Lemma 2.2.

$$F_{W_{n,r}}(x) - F_{W_{n+1,r}}(x) = \frac{f_{W_{n+1,r}}(x)}{2(1-x)}$$

Proof.

$$F_{W_{n,r}}(x) - F_{W_{n+1,r}}(x) = \Gamma_{-2\ln(1-x)}(n-1) - \Gamma_{-2\ln(1-x)}(n)$$
$$= \frac{1}{\Gamma(n)}(-2\ln(1-x))^{n-1}e^{-2\ln(1-x)}$$
$$= \frac{(1-x)^2}{\Gamma(n)}(-2\ln(1-x))^{n-1}$$
$$= \frac{f_{W_{n+1,r}}(x)}{2(1-x)}.$$
Using $p = 1$ and $p = 2$, we can get the mean and variance of $W_{nr}$ as

$$E(W_{nr}) = 1 - \left(\frac{2}{3}\right)^{n-1}$$

and

$$Var(W_{nr}) = \left(\frac{1}{2}\right)^{n-1} - \left(\frac{4}{9}\right)^{n-1}.$$

**Theorem 2.1.** Let $\mu_n^r = E(W_{nr}^r)$, then for $n \geq 2$ and $r = 1, 2, \ldots$

$$(r + 2)\mu_n^r - r\mu_n^{r-1} = 2\mu_{n-1}^r. \quad (2.2)$$

**Proof.**

$$r(\mu_n^{r-1} - \mu_n^r) = \frac{2^{n-1}}{\Gamma(n-1)\theta^2} \int_0^1 [(1-w)^2rw^{r-1}[-\ln(1-w)]^{n-2}dw$$

$$= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r[2(1-w)[-\ln(1-w)]^{n-2}dw$$

$$= \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1-w)^2w^{r-1}[-\ln(1-w)]^{n-3} \frac{1}{1-w}dw$$

$$= 2\mu_n^r - 2\mu_{n-1}^r.$$  

On simplification we get the result. □

**Theorem 2.2.** For $n \geq 2, p > 0$,

$$\mu_{nr}^p - \mu_{nr}^{p+1} = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}}$$
Proof.

\[
\mu_{pn}^r - \mu_{pn+1}^r = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1 - w) [-\ln(1 - w)]^{n-2} dw
\]

\[
= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p(1 - w)^2 [-\ln(1 - w)]^{n-2} dw
\]

\[
= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1 - w)^p w^2 [-\ln(w)]^{n-2} dw
\]

\[
= 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3 + k)^{n-1}} 
\]

(2.3)

Theorem 2.3. For \( n \geq 1 \),

\[
1 - W_{n+1r} \overset{d}{=} \Pi_{j=1}^n V_j, \quad (2.4)
\]

where \( V_1, V_2, ..., V_{n-1} \) are i.i.d. with \( F(v) = v^2, \ 0 < v < 1 \).

Proof. We will show first that

\[
1 - W_{n+1r} \overset{d}{=} (1 - W_{nr}) V_n,
\]

where \( V_n \) is independent of \( 1 - W_{nr} \) and is distributed with pdf as \( f_V(v) = 2v, \ 0 < v < 1 \).

Let \( Y_{n-1} = (1 - W_{nr}) V_n, \ n \geq 2, \) and \( f_n \) be the pdf of \( 1 - W_{nr} \), then

\[
f_n = \frac{2^{n-1} w}{\Gamma(n-1)} [-\ln(1 - w)]^{n-2}, \ 0 < w < 1,
\]

\[
F(y) = P(Y_{n+1} \leq y) = P(1 - W_{nr})V_n \leq y
\]

\[
= y^2 + \int_y^1 F_n(t) \frac{2}{t^2} dt, \text{ where } F_{n-1} \text{ is the df of } Y(n)
\]

\[
= y^2 + 2 \int_y^1 F_n(t) \frac{2}{t^2} dt
\]

\[
= y^2 + y^2 [F_n(t) \frac{1}{t^2}]_y + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt
\]

\[
= y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt
\]

\[
= F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt
\]

(2.5)
Differentiating both sides of (2.5), with respect to $y$, we obtain

$$f(y) = f_n(y) - f_n(y) + 2y \int_y^1 f_n(t) \frac{1}{t^2} dt$$

i.e.

$$\frac{f(y)}{y} = 2 \int_y^1 f_n(t) \frac{1}{t^2} dt$$

$$= 2 \int_y^1 \frac{2^{n-1}t}{\Gamma(n-1)} [-\ln(1-t)]^{n-2} \frac{1}{t^2} dt$$

$$= \frac{2^n}{\Gamma(n-1)} [-\ln t]^{n-1} \frac{1}{n-1} y$$

$$= \frac{2^n}{\Gamma(n)} [-\ln y]^{n-1}$$

(2.6)

Hence

$$f(y) = \frac{2^n}{\Gamma(n-1)} y [-\ln y]^{n-1}$$

(2.7)

which is the pdf of $Y = 1 - W_{n+1r}$.

Note that the sequence $Y_2, Y_3, \ldots$ forms a Markov chain. □

Using (2.6), we have the following representation of $W_{nr}$ for

$$1 - W_{n+1r} \overset{d}{=} \prod_{j=1}^n V_j, \quad n \geq 1$$

(2.8)

where $V_1, V_2, \ldots, V_{n-1}$ are i.i.d. with $F(v) = v^2, \quad 0 < v < 1$.

The conditional expectation of

$$1 - W_{nr} | 1 - W_{mr} = x, \quad 2 \leq m < n - 1, \quad 0 < x < 1,$$

is

$$E(1 - W_{nr} | 1 - W_{mr} = x) = x \left(\frac{2}{3}\right)^{n-m}.$$

Thus

$$Cov(W_{nr}, W_{mr}) = \left(\frac{2}{3}\right)^{n-m} Var(W_{mr})$$

$$= \left(\frac{2}{3}\right)^{n-m} \left[\left(\frac{1}{2}\right)^{m-1} - \left(\frac{4}{9}\right)^{m-1}\right].$$
The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$\rho_{m,n} = \frac{\left(\frac{2}{3}\right)^{n-m} \sqrt{\left[\left(\frac{1}{2}\right)^{m-1} - \left(\frac{1}{2}\right)^{n-1}\right]}}{\sqrt{\left[\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{n-1}\right]}} = \frac{\sqrt{\left(\frac{2}{3}\right)^{m-1} - 1}}{\sqrt{\left(\frac{2}{3}\right)^{n-1} - 1}} \rightarrow 0$$

for any fixed $m$ as $n \rightarrow \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \leq m \leq n = 5$.

| Table 2.1. variances and covariances of $W_{nr}$ and $W_{mr}$ |
|---|---|---|---|
| $m$ | $n$ | $2$ | $3$ | $4$ | $5$ |
| $2$ | $1/18$ | $1/27$ | $2/81$ | $4/243$ |   |
| $3$ | $1/27$ | $17/324$ | $17/386$ | $17/729$ |   |
| $4$ | $2/81$ | $17/386$ | $217/5382$ | $217/8748$ |   |
| $5$ | $4/243$ | $17/729$ | $217/8748$ | $2465/104976$ |   |

**Theorem 2.4.** The joint pdf $f_{m,n}^*$ of $W_{mr}$ and $W_{nr}$, $2 \leq m \leq n$, is given by

$$f_{m,n}^*(x, y) = \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{n-1}(-\ln(1-x))^{m-2} \times [-\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{(1-x)},$$

$0 < x < y < 1$.

**Proof.** Let $U_1 = \Pi_{j=1}^m V_j$ and $U_2 = \Pi_{j=1}^n V_{m+j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$f_{U_1U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m-1)}(-2 \ln u_1)^{m-2} \frac{2u_2}{\Gamma(n-m)}(-2 \ln u_2)^{n-m-1}.$$ 

Let $T_1 = U_1$ and $T_2 = U_1U_2$, then the joint pdf of $T_1$ and $T_2$ is

$$f_{T_1T_2}(t_1, t_2) = \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)}(-2 \ln t_1)^{m-2} t_2 \frac{2t_2}{t_1} (-2 \ln t_2)^{n-m-1}$$

$$= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln t_1)^{m-2} \frac{t_2}{t_1} (-\ln t_2 - \ln t_1)^{n-m-1}$$
Sustituting $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, We obtain the joint pdf of $W_{mr}$ and $W_{nr}$, $2 \leq m < n$ as

\[
f_{m,n}^*(x, y) = \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)} (-\ln(1 - x))^{m-2} \\
\times (-\ln(1 - x) + \ln(1 - y))^{n-m-1} \frac{1 - y}{(1 - x)},
\]

$0 < x < y < 1$. \[\square\]

**Theorem 2.5.** For $m \geq 2$, $p \geq 0$ and $q \geq 0$,

\[
E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q + 1}{q + 3} E([W_{mr}]^p(W_{m+1r})^q) + \frac{2}{q + 3} E(W_{mr})^{p+q+1}
\]

**Proof.**

\[
E([W_{mr}]^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1})
= \int_0^1 \int_x^1 [(x)^p \{(y)^q (1 - y)\}] f_{m,m+1}^*(x, y) dy dx
= \int_0^1 [(x)^p \frac{1}{\Gamma(m-1)} 2^{n-1} (-\ln(1 - x))^{m-2} \frac{1}{(1 - x)} H(x) dx, (2.9)
\]

where

\[
H(x) = \int_x^1 (y)^q (1 - y) (1 - y) dy
= \int_x^1 (y)^q (1 - y)^2 dy
= \frac{y^{q+1}}{q + 1} (1 - y)^2 \bigg|_x^1 + \int_x^1 \frac{2y^{q+1}}{q^{q+1} (q + 1)} (1 - y) dy
= -\frac{x^{q+1}}{q + 1} + 2 \int_x^1 \frac{y^{q+1}}{q + 1} (1 - y) dy
\]

Substituting in (2.9), we obtain

\[
E([W_{mr}]^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1})
= \int_0^1 [(x)^p (1 - y) \frac{1}{\Gamma(m-1)} 2^{m} (-\ln(1 - x))^{m-2} \frac{1}{(1 - x)}
\times \frac{2x^{q+1}}{q + 1} + 2 \int_x^1 \frac{y^{q+1}}{q + 1} (1 - y) dy] dx
= -\frac{2}{q + 1} E((W_{mr})^{p+q+1}) + \frac{2}{q + 1} E(W_{mr})^p(W_{m+1r})^{q+1}
\]
Thus

\[ E(W_{mr})^p(W_{m+1r})^{q+1} \]

\[ = \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q + 3} E(W_{mr})^{p+q+1}. \]

\[ \Box \]

**Theorem 2.6.** For \( m \geq 2, n > m \geq 2, p \geq 0 \) and \( q \geq 0 \),

\[ E(W_{mr})^p(W_{nr})^{q+1} \]

\[ = \frac{q + 1}{q + 3} E((W_{mr})^p(W_{nr})^q) + \frac{2}{q + 3} E((W_{mr})^p(W_{n+1r})^q) \]

\[ \text{Proof.} \]

\[ E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}] \]

\[ = \int_0^1 \int_0^1 [(x)^p(y)^q(1-y)]f^*_{m,n}(x,y)dydx \]

\[ = \int_0^1 [(x)^p \frac{1}{\Gamma(m-1)\Gamma(n-m)}2^{m+k-1}] \]

\[ (-\ln(1-x))^{m-2} \frac{1}{(1-x)}H(x)dx, \quad (2.10) \]

where

\[ \frac{1}{\Gamma(n-m)}H(x) \]

\[ = \int_x^1 [(y)^q(1-y)][-\ln(1-x) + \ln(1-y)]^{n-m-1} \]

\[ \times (1-y)dy \]

\[ = \int_x^1 \left[\frac{y}{q}\right]^q[-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2dy \]

\[ = \frac{y^{q+1}}{(q + 1)}[-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2\]

\[ - \int_x^1 \frac{y^{q+1}}{q + 1} dy \]

\[ \times (1-y)^2dy \]
\[= - \int_{x}^{1} \frac{y^{q+1}}{q+1} \frac{dy}{dy} \left[ - \ln(1 - x) + \ln(1 - y) \right]^{n-m-1} \times (1 - y)^2 \right] dy \]
\[= 2 \int_{x}^{1} \frac{y^{q+1}}{q+1} \left[ - \ln(1 - x) + \ln(1 - y) \right]^{n-m-1}(1 - y) \]
\[+ \int_{x}^{1} \frac{y^{q+1}}{\theta^2(q+1)} (n - m - 1) \left[ - \ln(1 - x) + \ln(1 - y) \right]^{n-m-2} \times (1 - \frac{y}{\theta}) \right] dy \]

Substituting \( H(x) \) in (2.10), we obtain
\[E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{nr})^{q+1}] \]
\[= \frac{2}{q+1} E((W_{mr})^p(W_{nr})^{q+1}) - \frac{1}{q+1} E((W_{mr})^p(W_{n-1r})^{q+1}) \]

On simplification, we obtain
\[E((W_{mr})^p(W_{nr})^{q+1}) \]
\[= \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q + 3} E((W_{mr})^p(W_{n-1r})^{q+1}) \square \]

**Entropy of** \( W_{nr} \).

The entropy of \( W_{nr} \) is given in the following theorem.

**Theorem 2.7.** The entropy, \( I_n \) of \( W_{nr} \), \( n \geq 2 \), is given by
\[I_n = \ln \Gamma(n - 1) + \frac{n - 1}{2} - (n - 2)\Psi(n - 1) - \ln 2, \]
where \( \Psi(n - 1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)} \).

**Proof.**
\[I_n = E(-\ln f_{W_{nr}}) \]
\[= \int_0^1 \left[ \ln \Gamma(n - 1) - (n - 1) \ln 2 - \ln(1 - u) - (n - 2) \ln(-\ln(1 - u)) \right] \frac{2^{n-1}(1 - u)}{\Gamma(n - 1)}[-\ln(1 - u)]^{n-2} du \]
\[= \ln \Gamma(n - 1) - (n - 1) \ln 2 - H_1 - H_2, \]
(2.11)
where

\[ H_1 = \int_0^1 \ln(1 - u) \frac{2^{n-1}(1 - u)}{\Gamma(n - 1)} \left( - \ln(1 - u) \right)^{n-2} = -\frac{n - 1}{2}, \]

\[ H_2 = \int_0^1 (n - 2) \ln(-\ln(1 - u)) \frac{2^{n-1}(1 - u)}{\Gamma(n - 1)} \left( - \ln(1 - u) \right)^{n-2} du \]

Substituting \(-\ln(1 - u) = t\), we obtain

\[ H_2 = \frac{2^{n-1}(n - 2)}{\Gamma(n - 1)} \int_0^\infty t^{n-2} \ln te^{-2t} dt \]

\[ = \frac{n - 2}{\Gamma(n - 1)} \int_0^\infty t^{n-2} \ln t e^{-2t} dt - (n - 2) \ln 2 \]

\[ = (n - 2) \left[ \Psi(n - 1) - \ln 2 \right]. \]

Substituting \(H_1\) and \(H_2\) in (2.11), we obtain

\[ I_n = \ln \Gamma(n - 1) - (n - 1) \ln 2 + \frac{n - 1}{2} - (n - 2) \left[ \Psi(n - 1) - \ln 2 \right]. \]

\[ = \ln \Gamma(n - 1) + \frac{n - 1}{2} - (n - 2) \Psi(n - 1) - \ln 2. \quad \square (2.12) \]

The following table gives \(-I_n\) for \(n = 3\) to \(10\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-I_n)</td>
<td>0.1159</td>
<td>0.3456</td>
<td>0.6698</td>
<td>1.0396</td>
<td>1.4363</td>
<td>1.8506</td>
<td>2.2775</td>
<td>2.7141</td>
</tr>
</tbody>
</table>

We will consider the estimation \(\theta\) based on the record range. when \(X_1, X_2, \ldots\) are i.i.d with \(f(x) = \frac{1}{\theta}, \ 0 < x < \theta\).

**Theorem 2.8.** The minimum variance linear unbiased estimator of \(\theta\) of \(\theta\) is

\[ \hat{\theta} = \frac{1}{3(2^n - 1)} \left[ 3.2^{n-1}W_{n+1r} - 2^{n-2}W_{nr} - 2^{n-3}W_{n-1r} - \ldots - W_2 \right] \]

and

\[ \text{Var}(\hat{\theta}) = \frac{\theta^2}{2(2^n - 1)}. \]
Proof. Let

\[
Z_1 = d_1 W_{2r}, \quad d_1 = 3.2^{\frac{1}{2}} \\
Z_2 = d_2 (W_{3r} - \frac{2}{3} W_{2r}), \quad d_2 = 3.2 \\
\vdots \\
Z_n = d_n (W_{n+1r} - \frac{2}{3} W_{nr}), \quad d_n = 3.2^{\frac{2}{n}} \\
Z' = (Z_1, Z_2, ..., Z_n),
\]

then \(E(Z') = A\theta\), where

\[
A' = (2^{\frac{1}{2}}, 2, ..., 2^{\frac{2}{n}}). A' A = 2(2^n - 1).
\]

Then the minimum variance linear unbiased estimator (MVLUE) \(\hat{\theta}\) of \(\theta\) (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is

\[
\hat{\theta} = (A' A)^{-1} A' Z
\]

\[
= \frac{1}{2(2^n - 1)} [2^{\frac{1}{2}} Z_1 + 2Z_2 + ... + 2^{\frac{2}{n}} Z_n]
\]

\[
= \frac{1}{2(2^n - 1)} [3.2^n W_{n+1r} - 2^{n-1} W_{nr} - 2^{n-2} W_{n-1r} - 2 W_{2r}]. \quad (2.13)
\]

\[
Var(\hat{\theta}) = \theta^2 (A' A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square
\]

For example, if \(n = 4\), then

\[
\hat{\theta} = \frac{1}{15} [24 W_{5r} - 4 W_{4r} - 2 W_{3r} - W_{2r}]
\]

and

\[
Var(\hat{\theta}) = \frac{\theta^2}{30}.
\]

Table 2.3. Coefficient of \(W_{nr}\) in MVLUE of \(\theta\).
Let $\tilde{\theta} = c\hat{\theta}$, then bias of $\tilde{\theta}$ is $(c - 1)\theta$ and mean squared error (MSE) of $\tilde{\theta}$ is $MSE(\tilde{\theta}) = c^2 \frac{\theta^2}{2n+1} + (c - 1)^2 \theta^2$. The MSE of $\tilde{\theta}$ will be minimum if $c = \frac{2n+1-2}{2n+1-1}$.

The bias of $\tilde{\theta} = (c - 1)\theta = \frac{1}{2n+1}$ and $MSE(\tilde{\theta}) = \frac{1}{2n+1}$.

**Prediction of $W_{n+sr}$.**

We consider the prediction of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$.

**Theorem 2.5.** The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is $\theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr}$.

**Proof.** The best linear least squares predictor, $W^*_{n+sr}$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is

$$W^*_{n+sr} = E(W_{n+sr}|W_{2r} = x_2, W_{3r} = x_3, \ldots, W_{nr} = x_n) = E(W_{n+sr}|W_{nr} = x_n), \text{ by Markov property of } W_{2r}, W_{3r}, \ldots = \theta[1 - (\frac{2}{3})^s] + x_n(\frac{2}{3})^s.$$  

**References**

