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Record Range of Uniform Distribution

M. Ahsanullah

Department of Management Sciences, Rider University, Lawrenceville, NJ, USA.

Abstract. We consider a sequence of independent and identicaly distributed (iid) random variables with absolutely continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let R_{nl} be the largest observation after observing nth record and $R_{(ns)}$ be the smallest observation after observing the nth record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the nth record range. We will consider some distributional properties of W_{nr} when $f(x) = 1$, $0 \leq x \leq 1$.

1 Introduction

Let $\{X_i, i = 1, 2, ..\}$ be a sequence of independent and identically distributed random varibles with an absolutely continuous(with respect to Lebegue measure) ditribution function F(x) with pdf f(x).Let $\mathrm{R}_{U(1)} = X_1$, $\mathrm{R}_{U(2)}, ...$,be the upper records and $\mathrm{R}_{L(1)}, R_{L(2)}, ...$be lower records of $\{\mathrm{X}_i, i = 1, 2, ..\}$.For various properties of record values see Ahsanullah(2005) Arnold et. al.(1998).

Suppose R_{nl} is the largest observation after observing nth record and $\mathrm{R}_{(ns)}$is the smallest observation after observing the nth record. Then we say $\mathrm{W}_{nr} = R_{nl} - R_{(ns)}, n > 1$,as the nth record range.The

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joint pdf of $f_{(nl,(ns))}$ of R_{nl} and $R_{(ns)}$ is given by (see Arnold et. al. 1998, p. 275) as

$$f_{(nl,(ns))}(x,y) = \frac{2^{n-1}}{(n-2)!}\,[-\ln(\overline{F}(y)+F(x))]^{n-2} f(x)\ f(y), \quad (1.1)$$

$$-\infty < x < y < \infty$$

The pdf of $f_{W_{nr}}$ of W_{nr} is given by

$$f_{W_{nr}}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!}\,[-\ln(\overline{F}(w+u)+F(u))]^{n-2} f(w+u)\ f(u)\ du \quad (1.2)$$

Suppose $X_i's$ are distributed as uniform with

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

Using (1.3) in (1.2), we obtain

$$f_{W_{nr}}(w) = \begin{cases} \frac{2^{n-1}(1-w)}{\Gamma(n-1)}[-\ln\{1-w)]^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases} \quad (1.4)$$

Figure 1.1 gives the pdf of W_{nr} for $n = 10$ when X_i' are distributed as uniform.

In this paper we will consider distributional properties of W_{nr} for the case $X_i's$ are distributed as uniform distribution.

2 Main Results

Lemma 2.1. *For $n \geq 2$ and $0 < x < 1$,*

$$F_{W_{nr}}(x) = \Gamma_{-2\ln(1-x)}(n-1),$$

where

$$\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du$$

Proof.

$$\begin{aligned} F_{W_{nr}}(x) &= \int_0^x \frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln(1-u)]^{n-2} du \\ &= \int_0^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} dt \\ &= \Gamma_{-2\ln(1-x)}(n-1). \ \square \end{aligned}$$

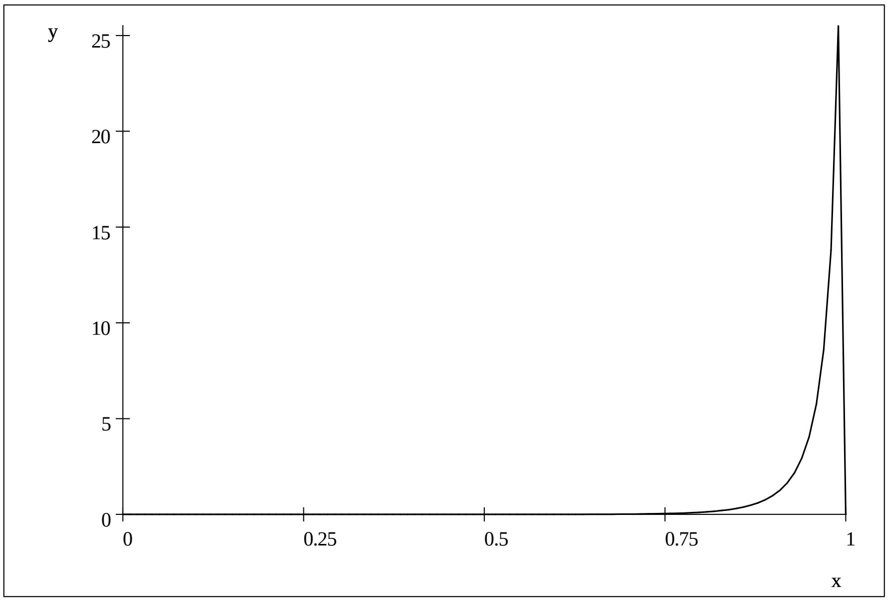

Figure 1.1: $f(x) = \frac{2^9(1-x)}{\Gamma(9)}(-\ln(1-x))^8$

Remark 2.1.

$$1 - F_{W_{nr}}(x) = (1-x)^2 \sum_{j=0}^{n-2} \frac{(-2\ln(1-x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2$$

Lemma 2.2.

$$F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1-x)}$$

Proof.

$$\begin{aligned} F_{W_{nr}}(x) - F_{W_{n+1r}}(x) &= \Gamma_{-2\ln(1-x)}(n-1) - \Gamma_{-2\ln(1-x)}(n) \\ &= \frac{1}{\Gamma(n)}(-2\ln(1-x))^{n-1} e^{-2\ln(1-x)} \\ &= \frac{(1-x)^2}{\Gamma(n)}(-2\ln(1-x))^{n-1} \\ &= \frac{f_{W_{n+1r}}(x)}{2(1-x)}. \end{aligned}$$

$$\begin{aligned}\mu_{nr}^{p} &= E(W_{nr})^{p} = \frac{2^{n-1}}{(n-2)!}\int_0^1 w^p(1-w)\ [-\ln(1-u)]^{n-2}dw \\ &= \frac{2^{n-1}(n-2)!}{(n-2)!}\int_0^1 (1-w)^p w\ [-\ln(w)]^{n-2}dw \\ &= 2^{n-1}(\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k}{(2+k)^{n-1}} \qquad (2.1)\end{aligned}$$

Using $p=1$ and $p=2$, we can get the mean and variance of W_{nr} as

$$E(W_{nr})\ = 1-(\frac{2}{3})^{n-1}$$

and

$$Var(W_{nr})\ = (\frac{1}{2})^{n-1}-(\frac{4}{9})^{n-1}.\ \square$$

Theorem 2.1. *Let* $\mu_n^r = E(W_{nr}^r)$*, then for* $n \geq 2$ *and* $r = 1, 2, \dots$

$$(r+2)\mu_n^r - r\mu_n^{r-1} = 2\mu_{n-1}^r. \qquad (2.2)$$

Proof.

$$\begin{aligned}& r(\mu_n^{r-1}-\mu_n^r) \\ &= \frac{2^{n-1}}{\Gamma(n-1)\theta^2}\int_0^1 [(1-w)^2 r w^{r-1}[-\ln(1-w)]^{n-2}dw \\ &= \frac{2^{n-1}}{\Gamma(n-1)}\int_0^1 w^r[2(1-w)[-\ln(1-w)]^{n-2}dw \\ &-\frac{2^{n-1}(n-2)}{\Gamma(n-1)}\int_0^1 (1-w)^2 w^r[-\ln(1-w)]^{n-3}\frac{1}{1-w}dw \\ &= 2\mu_n^r - 2\mu_{n-1}^r.\end{aligned}$$

On simplification we get the result. $\square$

Theorem 2.2. *For* $n \geq 2, p > 0$,

$$\mu_{nr}^{p} - \mu_{nr}^{p+1} = 2^{n-1}\sum_{k=0}^{p}\binom{p}{k}\frac{(-1)^k}{(3+k)^{n-1}}$$

Proof.

$$\begin{aligned}
\mu^{p}_{nr} - \mu^{p+1}_{nr} &= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1-w)\ [-\ln(1-w)]^{n-2} dw \\
&= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p (1-w)^2\ [-\ln(1-w)]^{n-2} dw \\
&= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1-w)^p\ w^2 [-\ln(w)]^{n-2} dw \\
&= 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \qquad (2.3)
\end{aligned}$$

Theorem 2.3. *For $n \geq 1$,*

$$1 - W_{n+1r} \stackrel{d}{=} \Pi_{j=1}^{n} V_j, \qquad (2.4)$$

where $V_1, V_2, ..., V_{n-1}$ are i.i.d. with $F(v) = v^2$, $0 < v < 1$.

Proof. We will show first that

$$1 - W_{n+1r} \stackrel{d}{=} (1 - W_{nr}) V_n,$$

where V_n, is indendent of $1 - W_{nr}$ and is distributed with pdf as $f_V(v). = 2v$, $0 < v < 1$.

Let $Y_{n-1} = (1 - W_{nr}) V_n$, $n \geq 2$, and f_n be the pdf of $1 - W_{nr}$, then

$$f_n = \frac{2^{n-1} w}{\Gamma(n-1)} [-\ln(1-w)]^{n-2}, \quad 0 < w < 1,$$

$$\begin{aligned}
F(y) &= P(Y_{n+1} \leq y) = P(1 - W_{nr}) V_n \leq y) \\
&= y^2 + \int_y^1 F_n(\frac{y}{v}) 2v dv, \text{ where } F_{n-1} \text{ is the df of } Y_{(n)} \\
&= y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^3} dt \\
&= y^2 + y^2 [F_n(t) \frac{-1}{t^2}]_y^1 + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
&= y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
&= F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \qquad (2.5)
\end{aligned}$$

Differentiang both sides of (2.5), with respect to y, we obtain

$$\begin{aligned} f(y) &= f_n(y) - f_n(y) + 2y\int_y^1 f_n(t)\frac{1}{t^2}dt \\ &= 2y\int_y^1 f_n(t)\frac{1}{t^2}dt \end{aligned}$$

i.e.

$$\begin{aligned} \frac{f(y)}{y} &= 2\int_y^1 f_n(t)\frac{1}{t^2}dt \\ &= 2\int_y^1 \frac{2^{n-1}t}{\Gamma(n-1)}[-\ln(1-t]^{n-2}\frac{1}{t^2}dt \\ &= [\frac{2^n}{\Gamma(n-1)}[-\ln t]^{n-1}\frac{-1}{n-1}]_y^1 \\ &= \frac{2^n}{\Gamma(n)}[-\ln y]^{n-1} \end{aligned} \tag{2.6}$$

Hence

$$f(y) = \frac{2^n}{\Gamma(n-1)}y[-\ln y]^{n-1} \tag{2.7}$$

which is the pdf of $Y = 1 - W_{n+1r}$.

Note that the sequence $Y_2, Y_3, ...$forms a Markov chain. □

Using (2.6), we have the following representation of W_{nr} for

$$1 - W_{n+1r} \stackrel{d}{=} \Pi_{j=1}^n V_j, \quad n \geq 1 \tag{2.8}$$

where $V_1, V_2, ..., V_{n-1}$ are i.i.d. with $F(v) = v^2, \quad 0 < v < 1$.

The conditional expectation of

$$1 - W_{nr}|1 - W_{mr} = x, \ 2 \leq m < n-1, \ 0 < x < 1,$$

is

$$E(1 - W_{nr}|1 - W_{mr} = x) = x(\frac{2}{3})^{n-m}.$$

Thus

$$\begin{aligned} Cov(W_{nr}W_{mr}) &= (\frac{2}{3})^{n-m}Var(W_{mr}) \\ &= (\frac{2}{3})^{n-m}\ [(\frac{1}{2})^{m-1} - (\frac{4}{9})^{m-1}]. \end{aligned}$$

The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$\rho_{m,n} = \frac{(\frac{2}{3})^{n-m}\sqrt{[(\frac{1}{2})^{m-1} - (\frac{4}{9})^{m-1}]}}{\sqrt{[(\frac{1}{2})^{n-1} - (\frac{4}{9})^{n-1}]}} = \frac{\sqrt{[(\frac{9}{8})^{m-1} - 1]}}{\sqrt{[(\frac{9}{8})^{n-1} - 1]}} \longrightarrow 0$$

for any fixed m as $n \longrightarrow \infty$.

The following table gives the variances and covariances of W_{nr} and W_{mr} for $2 \le m \le n = 5$.

Table 2.1. variances and covariances of W_{nr} and W_{mr}

		m			
		2	3	4	5
	2	$\frac{1}{18}$	$\frac{1}{27}$	$\frac{2}{81}$	$\frac{4}{243}$
n	3	$\frac{1}{27}$	$\frac{17}{324}$	$\frac{17}{486}$	$\frac{17}{729}$
	4	$\frac{2}{81}$	$\frac{17}{486}$	$\frac{217}{5862}$	$\frac{217}{8748}$
	5	$\frac{4}{243}$	$\frac{17}{729}$	$\frac{217}{8748}$	$\frac{2465}{104976}$

Theorem 2.4. *The joint pdf $f^*_{m,n}$ of W_{mr} and W_{nr}, $2 \le m \le n$, is given by*

$$\begin{aligned} f^*_{m,n}(x,y) &= \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{n-1}(-\ln(1-x))^{m-2} \\ &\times [-\ln(1-x) + \ln(1-y)]^{n-m-1}\frac{1-y}{(1-x)}, \\ & 0 < x < y < 1. \end{aligned}$$

Proof. Let $U_1 = \Pi_{j=1}^{m} V_j$ and $U_2 = \Pi_{j=1}^{n-m} V_{m+j}$, then the joint pdf of U_1 and U_2 is given by

$$f_{U_1U_2}(u_1,u_2) = \frac{2u_1}{\Gamma(m-1)}(-2\ln u_1)^{m-2}\frac{2u_2}{\Gamma(n-m)}(-2\ln u_2)^{n-m-1}.$$

Let $T_1 = U_1$ and $T_2 = U_1U_2$, then the joint pdf of T_1 and T_2 is

$$\begin{aligned} & f_{T_1T_2}(t_1,t_2) \\ &= \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)}(-2\ln t_1)^{m-2} 2\frac{t_2}{t_1^2}\ (-2\ln(\frac{t_2}{t_1}))^{n-m-1} \\ &= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln t_1)^{m-2}\frac{t_2}{t_1}\ (-(\ln t_2 - \ln t_1))^{n-m-1} \end{aligned}$$

Sustituting $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, We obtain the joint pdf of W_{mr} and W_{nr}, $2 \le m < n$ as

$$\begin{aligned} f^*_{m,n}(x,y) &= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln(1-x))^{m-2} \\ &\times \; [-\ln(1-x)+\ln(1-y)]^{n-m-1}\frac{1-y}{(1-x)}, \\ & \quad 0 < x < y < 1. \; \Box \end{aligned}$$

Theorem 2.5. *For $m \ge 2$, $p \ge 0$ and $q \ge 0$,*

$$E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q+1}{q+3}E[(W_{mr})^p(W_{m+1r})^q + \frac{2}{q+3}E(W_{mr})^{p+q+1}$$

Proof.

$$\begin{aligned} & E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] \\ &= \int_o^1\int_x^1 [(x)^p\{(y)^q(1-y)\}]f^*_{m,m+1}(x,y)dydx \\ &= \int_o^\theta [(x)^p\frac{1}{\Gamma(m-1)}2^{n-1}(-\ln(1-x))^{m-2}\frac{1}{(1-x)}H(x)dx, \quad (2.9) \end{aligned}$$

where

$$\begin{aligned} H(x) &= \int_x^1 (y)^q(1-y)\}(1-y)dy \\ &= \int_x^1 (y)^q(1-y)^2dy \\ &= \frac{y^{q+1}}{(q+1)}(1-y)^2|_x^1 + \int_x^1 \frac{2y^{q+1}}{\theta^{q+1}(q+1)}(1-y) \\ &= -\frac{x^{q+1}}{(q+1)} + 2\int_x^1 \frac{y^{q+1}}{(q+1)}(1-y)dy \end{aligned}$$

Substituting in (2.9), we obtain

$$\begin{aligned} & E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] \\ &= \int_o^1 [(x)^p(1-y)\frac{1}{\Gamma(m-1)}2^m(-\ln(1-x))^{m-2}\frac{1}{(1-x)} \\ & \quad .[-\frac{2x^{q+1}}{(q+1)} + 2\int_x^1 \frac{y^{q+1}}{(q+1)}(1-y)dy]dx \\ &= -\frac{2}{q+1}E((W_{mr})^{p+q+1}) + \frac{2}{q+1}E(W_{mr})^p(W_{m+1r})^{q+1} \end{aligned}$$

Thus

$$E(W_{mr})^p(W_{m+1r})^{q+1}$$
$$= \frac{q+1}{q+3}E[(W_{mr})^p(W_{m+1r})^q + \frac{2}{q+3}E(W_{mr})^{p+q+1}. \quad \square$$

Theorem 2.6. *For $m \geq 2$, $n > m \geq 2$, $p \geq 0$ and $q \geq 0$,*

$$E(W_{mr})^p(W_{nr})^{q+1}$$
$$= \frac{q+1}{q+3}E((W_{mr})^p(W_{nr})^q) + \frac{2}{q+3}E(W_{mr})^{p+q+1}$$
$$E((W_{mr})^p(W_{nr})^{q+1})$$
$$= \frac{q+1}{q+3}E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3}E((W_{mr})^p(W_{n-1r})^q)$$

Proof.

$$E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}]$$
$$= \int_o^1\int_x^1 [(x)^p\{(y)^q(1-y)\}]f^*_{m,n}(x,y)dydx$$
$$= \int_o^1 [(x)^p \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{m+k-1}$$
$$(-\ln(1-x))^{m-2}\frac{1}{(1-x)}H(x)dx, \qquad (2.10)$$

where

$$\frac{1}{\Gamma(n-m)}H(x)$$
$$= \int_x^1 \{(y)^q(1-y)\}[-\ln(1-x)+\ln(1-y)]^{n-m-1}$$
$$\times(1-y)dy$$
$$= \int_x^1 (\frac{y}{\theta})^q[-\ln(1-x)+\ln(1-y)]^{n-m-1}(1-y)^2dy$$
$$= \frac{y^{q+1}}{(q+1)}[-\ln(1-x)+\ln(1-y)]^{n-m-1}(1-y)^2]_x^1$$
$$- \int_x^1 \frac{y^{q+1}}{q+1}\frac{d}{dy}[-\ln(1-x)+\ln(1-y)]^{n-m-1}$$
$$\times(1-y)^2dy$$

$$= -\int_x^1 \frac{y^{q+1}}{q+1}\frac{d}{dy}[-\ln(1-x)+\ln(1-y)]^{n-m-1}$$

$$\times(1-y)^2]dy$$

$$= 2\int_x^1 \frac{y^{q+1}}{q+1}[-\ln(1-x)+\ln(1-y)]^{n-m-1}(1-y)]$$

$$+\int_x^1 \frac{y^{q+1}}{\theta^q(q+1)}(n-m-1)[-\ln(1-x)+\ln(1-y)]^{n-m-2}$$

$$\times(1-\frac{y}{\theta})dy$$

Substituting H(x) in (2.10), we obtain

$$E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{nr})^{q+1}]$$
$$= \frac{2}{q+1}E((W_{mr})^p(W_{nr})^{q+1}) - \frac{1}{q+1}E((W_{mr})^p(W_{n-1r})^{q+1})$$

On simplification, we obtain

$$E((W_{mr})^p(W_{nr})^{q+1})$$
$$= \frac{q+1}{q+3}E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3}E((W_{mr})^p(W_{n-1r})^{q+1}) \quad \square$$

Entropy of W_{nr}.
The entropy of W_{nr} is given in the following theorem.

Theorem 2.7. *The entropy, I_n of W_{nr}, $n \geq 2$, is given by*

$$I_n = \ln\Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2,$$

where $\Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)}$.
Proof.

$$\begin{aligned} I_n &= E(-\ln f_{W_{nr}}) \\ &= \int_0^1 [\ln\Gamma(n-1) - (n-1)\ln 2 - \ln(1-u) - (n-2) \\ &\quad \ln(-\ln(1-u))]\frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln\{1-u)]^{n-2}du \\ &= [\ln\Gamma(n-1) - (n-1)\ln 2 - H_1 - H_2, \end{aligned} \quad (2.11)$$

where

$$\begin{aligned} H_1 &= \int_0^1 \ln(1-u)\frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln\{1-u)]^{n-2} = -\frac{n-1}{2}, \\ H_2 &= \int_0^1 (n-2)ln(-ln(1-u))]\frac{2^{n-1}(1-u)}{\Gamma(n-1)}[-\ln\{1-u)]^{n-2}du \end{aligned}$$

Substituing $-\ln\{1-u)=t$, we obtain

$$\begin{aligned} H_2 &= \frac{2^{n-1}(n-2)}{\Gamma(n-1)}\int_0^\infty t^{n-2}\ln te^{-2t}dt \\ &= \frac{n-2}{\Gamma(n-1)}\int_0^\infty t^{n-2}\ln te^{-t}dt-(n-2)\ln 2 \\ &= (n-2)[\Psi(n-1)-\ln 2]. \end{aligned}$$

Substituting H_1 and H_2 in (2.11), we obtain

$$\begin{aligned} I_n &= \ln\Gamma(n-1)-(n-1)\ln 2+\frac{n-1}{2} \\ &\quad -(n-2)[\Psi(n-1)-\ln 2]. \\ &= \ln\Gamma(n-1)+\frac{n-1}{2}-(n-2)\Psi(n-1)-\ln 2. \quad \Box(2.12) \end{aligned}$$

The following table gives $-I_n$ for $n=3$ to 10.

Table 2.2. Values of $-I_n$ for $2\le n\le 10$.

n	3	4	5	6	7	8	9	10
-I$_n$	0.1159	0.3456	0.6698	1.0396	1, 4363	1.8506	2.2775	2.7141

We will consider the estimation θ based on the record range. when $X_1, X_2, \ldots$ are i.i.d with $f(x)=\frac{1}{\theta},\ 0<x<\theta$.

Theorem 2.8. *The minimum varaiance linear unbiased estimator of $\hat\theta$ of θ is*

$$\hat\theta = \frac{1}{3(2^n-1)}[3.2^{n-1}W_{n+1r}-2^{n-2}W_{nr}-2^{n-3}W_{n-1r}-...-W_{2r}]$$

and

$$Var(\hat\theta)=\frac{\theta^2}{2(2^n-1)}.$$

Proof. Let

$$\begin{aligned} Z_1 &= d_1 W_{2r}, d_1 = 3.2^{\frac{1}{2}} \\ Z_2 &= d_2(W_{3r} - \frac{2}{3}W_{2r}), d_2 = 3.2 \\ &\vdots \\ Z_n &= d_n(W_{n+1r} - \frac{2}{3}W_{nr}), d_n = 3.2^{\frac{n}{2}}. \\ Z' &= (Z_1, Z_2,)..., Z_n), \end{aligned}$$

then $E(Z') = A\theta$, where

$$A' = (2^{\frac{1}{2}}, 2, ..., 2^{\frac{n}{2}}).A'A = 2(2^n - 1).$$

Then the minimum variance linear unbiased estimator (MVLUE) $\hat{\theta}$ of θ (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is

$$\begin{aligned} \hat{\theta} &= (A'A)^{-1}A'Z \\ &= \frac{1}{2(2^n-1)}[2^{\frac{1}{2}}Z_1 + 2Z_2 + ... + 2^{\frac{n}{2}}Z_n] \\ &= \frac{1}{2(2^n-1)}[3.2^n W_{n+1r} - 2^{n-1}W_{nr} - 2^{n-2}W_{n-1r} - 2W_{2r}]. \quad (2.13) \end{aligned}$$

$$Var(\hat{\theta}) = \theta^2 (A'A)^{-1} = \frac{\theta^2}{2(2^n-1)}. \quad \square$$

For example, if $n = 4$, then

$$\hat{\theta} - \frac{1}{15}[24W_{5r} - 4W_{4r} - 2W_{3r} - W_{2r}]$$

and

$$Var(\hat{\theta}) = \frac{\theta^2}{30}.$$

Table 2.3. Coefficient of W_{nr} in MVLUE of θ.

n	W_{2r}	W_{3r}	W_{4r}	W_{5r}	W_{6r}	W_{7r}	W_{8r}	W_{9r}	W_{10r}
2	$-\frac{1}{3}$	2							
3	$-\frac{1}{7}$	$-\frac{2}{7}$	$\frac{12}{7}$						
4	$-\frac{1}{15}$	$-\frac{2}{15}$	$-\frac{4}{15}$	$\frac{24}{15}$					
5	$-\frac{1}{31}$	$-\frac{2}{31}$	$-\frac{4}{31}$	$-\frac{8}{31}$	$\frac{48}{31}$				
6	$-\frac{1}{63}$	$-\frac{2}{63}$	$-\frac{4}{63}$	$-\frac{8}{63}$	$-\frac{16}{63}$	$\frac{96}{63}$			
7	$-\frac{1}{127}$	$-\frac{2}{127}$	$-\frac{4}{127}$	$-\frac{8}{127}$	$-\frac{16}{127}$	$-\frac{32}{127}$	$\frac{192}{127}$		
8	$-\frac{1}{255}$	$-\frac{2}{299}$	$-\frac{4}{255}$	$-\frac{8}{255}$	$-\frac{16}{255}$	$-\frac{32}{255}$	$-\frac{64}{255}$	$\frac{384}{255}$	
9	$-\frac{1}{511}$	$-\frac{2}{511}$	$-\frac{4}{511}$	$-\frac{8}{311}$	$-\frac{16}{511}$	$-\frac{32}{511}$	$-\frac{64}{511}$	$-\frac{128}{511}$	$\frac{768}{511}$

Let $\tilde{\theta} = c\hat{\theta}$, then bias of $\tilde{\theta}$ is $(c-1)\theta$ and mean squared error (MSE) of $\tilde{\theta}$ is $\text{MSE}(\tilde{\theta}) = c^2\frac{\theta^2}{2(2^n-1)} + (c-1)^2\theta^2$. The MSE of $\tilde{\theta}$ will be minimum if $c = \frac{2^{n+1}-2}{2^{n+1}-1}$.

The bias of $\tilde{\theta} = (c-1)\theta = \frac{-1}{2^{n+1}-1}$ and $\text{MSE}(\tilde{\theta}) = \frac{1}{2^{n+1}-1}$.

Prediction of W_{n+sr}.
We consider the prediction of W_{n+sr} based on $W_{2r}, W_{3r}, \ldots, W_{nr}$.

Theorem 2.5. *The best linear least squares predictor, W^*_{n+sr} of W_{n+sr} based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is $\theta[1-(\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr}$.*

Proof. The best linear least squares predictor, W^*_{n+sr} of W_{n+sr} based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is

$$\begin{aligned} W^*_{n+sr} &= E(W_{n+sr}|W_{2r}=x_2, W_{3r}=x_3, ..., W_{nr}=x_n) \\ &= E(W_{n+sr}|W_{nr}=x_n), \text{by Markov property of } W_{2r}, W_{3r}, \ldots \\ &= \theta[1-(\frac{2}{3})^s] + x_n(\frac{2}{3})^s. \end{aligned}$$

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