Record Range of Uniform Distribution

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Abstract. We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $R_{nl}$ be the largest observation after observing $n$th record and $R_{(ns)}$ be the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range. We will consider some distributional properties of $W_{nr}$ when $f(x) = 1, 0 \leq x \leq 1$.

1 Introduction

Let $\{X_i, i = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebegue measure) distribution function $F(x)$ with pdf $f(x)$. Let $R_{U(1)} = X_1, R_{U(2)}, \ldots$, be the upper records and $R_{L(1)}, R_{L(2)}, \ldots$ be lower records of $\{X_i, i = 1, 2, \ldots\}$. For various properties of record values see Ahsanullah (2005) Arnold et. al. (1998).

Suppose $R_{nl}$ is the largest observation after observing $n$th record and $R_{(ns)}$ is the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range.

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joint pdf of $f_{(nl,(ns))}$ of $R_{nl}$ and $R_{(ns)}$ is given by (see Arnold et. al. 1998, p. 275) as

$$f_{(nl,(ns))}(x,y) = \frac{2^{n-1}}{(n-2)!} \left(-\ln(F(y) + F(x))\right)^{n-2} f(x) f(y), \quad (1.1)$$

$-\infty < x < y < \infty$

The pdf of $f_{wnr}$ of $W_{nr}$ is given by

$$f_{wnr}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} \left[-\ln(F(w+u)+F(u))\right]^{n-2} f(w+u) f(u) du \quad (1.2)$$

Suppose $X'_i$'s are distributed as uniform with

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

Using (1.3) in (1.2), we obtain

$$f_{wnr}(w) = \begin{cases} 2^{n-1}(1-w) \left[-\ln(1-w)\right]^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases} \quad (1.4)$$

Figure 1.1 gives the pdf of $W_{nr}$ for $n = 10$ when $X'_i$ are distributed as uniform.

In this paper we will consider distributional properties of $W_{nr}$ for the case $X'_i$'s are distributed as uniform distribution.

## 2 Main Results

**Lemma 2.1.** For $n \geq 2$ and $0 < x < 1$,

$$F_{W_{nr}}(x) = \Gamma_{-2\ln(1-x)}(n-1),$$

where

$$\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du$$

**Proof.**

$$F_{W_{nr}}(x) = \int_0^x \frac{2^{n-1}(1-u)}{\Gamma(n-1)} \left[-\ln(1-u)\right]^{n-2} du$$

$$= \int_0^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} dt$$

$$= \Gamma_{-2\ln(1-x)}(n-1). \quad \square$$
Remark 2.1.

\[ 1 - F_{W_{nr}}(x) = (1 - x)^2 \sum_{j=0}^{n-2} \frac{(-2 \ln(1 - x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2 \]

Lemma 2.2.

\[ F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1 - x)} \]

Proof.

\[
F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \Gamma_{-2 \ln(1-x)}(n-1) - \Gamma_{-2 \ln(1-x)}(n) \\
= \frac{1}{\Gamma(n)}(-2 \ln(1-x))^{n-1} e^{-2 \ln(1-x)} \\
= \frac{(1 - x)^2}{\Gamma(n)}(-2 \ln(1-x))^{n-1} \\
= \frac{f_{W_{n+1r}}(x)}{2(1 - x)}. \]
\[ \mu_{nr}^p = E(W_{nr})^p = \frac{2^{n-1}}{(n-2)!} \int_0^1 w^p(1-w) \left[ -\ln(1-w) \right]^{n-2} dw \]
\[ = \frac{2^{n-1}(n-2)!}{(n-2)!} \int_0^1 (1-w)^p w \left[ -\ln(w) \right]^{n-2} dw \]
\[ = 2^{n-1} \left( \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(2+k)^{n-1}} \right) \tag{2.1} \]

Using \( p = 1 \) and \( p = 2 \), we can get the mean and variance of \( W_{nr} \) as

\[ E(W_{nr}) = 1 - \left( \frac{2}{3} \right)^{n-1} \]

and

\[ Var(W_{nr}) = \left( \frac{1}{2} \right)^{n-1} - \left( \frac{4}{9} \right)^{n-1}. \]

**Theorem 2.1.** Let \( \mu_n^r = E(W_{nr}^r) \), then for \( n \geq 2 \) and \( r = 1, 2, \ldots \)

\[ (r+2)\mu_n^r - r\mu_{n-1}^r = 2\mu_{n-1}^r. \tag{2.2} \]

**Proof.**

\[ r(\mu_n^{r-1} - \mu_n^r) \]
\[ = \frac{2^{n-1}}{\Gamma(n-1)\theta^2} \int_0^1 [(1-w)^2rw^{r-1}[-\ln(1-w)]^{n-2} dw \]
\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r[2(1-w)[-\ln(1-w)]^{n-2} dw \]
\[ - \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1-w)^2w^r[-\ln(1-w)]^{n-3} \frac{1}{1-w} dw \]
\[ = 2\mu_n^r - 2\mu_{n-1}^r. \]

On simplification we get the result. \( \square \)

**Theorem 2.2.** For \( n \geq 2, p > 0, \)

\[ \mu_{nr}^p - \mu_{nr}^{p+1} = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \]
Proof.

\[ \mu_{n+1} - \mu_n = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^{n+1} - w^n)(1 - w) \left[-\ln(1 - w)\right]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^n(1 - w)^2 \left[-\ln(1 - w)\right]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1 - w)^n w^2 [-\ln(w)]^{n-2} dw \]

\[ = 2^{n-1} \sum_{k=0}^{\infty} \frac{(9k)}{(3 + k)^n} \]  \hspace{1cm} (2.3)

Theorem 2.3. For \( n \geq 1 \),

\[ 1 - W_{n+1} \overset{d}{=} \Pi_{j=1}^n V_j, \]  \hspace{1cm} (2.4)

where \( V_1, V_2, ..., V_{n-1} \) are i.i.d. with \( F(v) = v^2 \), \( 0 < v < 1 \).

Proof. We will show first that

\[ 1 - W_{n+1} \overset{d}{=} (1 - W_n)V_n, \]

where \( V_n \) is independent of \( 1 - W_n \) and is distributed with pdf as \( f_V(v) = 2v \), \( 0 < v < 1 \).

Let \( Y_{n-1} = (1 - W_n)V_n \), \( n \geq 2 \), and \( f_n \) be the pdf of \( 1 - W_n \), then

\[ f_n = \frac{2^{n-1}w}{\Gamma(n-1)}[-\ln(1 - w)]^{n-2}, \quad 0 < w < 1, \]

\[ F(y) = P(Y_{n+1} \leq y) = P(1 - W_n)V_n \leq y) \]

\[ = y^2 + \int_y^1 F_n(t) \frac{2v}{t^3} dt, \text{ where } F_{n-1} \text{ is the df of } Y_{(n)} \]

\[ = y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^3} dt \]

\[ = y^2 + y^2[F_n(t) \frac{1}{t^2}]_y + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]  \hspace{1cm} (2.5)
Differentiating both sides of (2.5), with respect to \( y \), we obtain

\[
f(y) = f_n(y) - f_n(y) + 2y \int_y^1 f_n(t) \frac{1}{t^2} dt
\]

i.e.

\[
\frac{f(y)}{y} = 2 \int_y^1 f_n(t) \frac{1}{t^2} dt
\]

\[
= 2 \int_y^1 \frac{2^{n-1} t^{n-2}}{\Gamma(n-1)} \left[-\ln(1-t)^{n-2} \frac{1}{t^2} dt\right]
\]

\[
= \left[ \frac{2^n}{\Gamma(n-1)} \left[-\ln t\right]^{n-1} \right]
\]

\[
= \frac{2^n}{\Gamma(n)} \left[-\ln y\right]^{n-1}
\]

Hence

\[
f(y) = \frac{2^n}{\Gamma(n-1)} y \left[-\ln y\right]^{n-1}
\]

(2.6)

which is the pdf of \( Y = 1 - W_{n+1r} \).

Note that the sequence \( Y_2, Y_3, \ldots \) forms a Markov chain. □

Using (2.6), we have the following representation of \( W_{nr} \) for

\[
1 - W_{n+1r} \overset{d}{=} \prod_{j=1}^{r} V_j, \quad n \geq 1
\]

(2.8)

where \( V_1, V_2, \ldots, V_{n-1} \) are i.i.d. with \( F(v) = v^2, \quad 0 < v < 1 \).

The conditional expectation of

\[
1 - W_{nr} | 1 - W_{mr} = x, \quad 2 \leq m < n - 1, \quad 0 < x < 1,
\]

is

\[
E(1 - W_{nr} | 1 - W_{mr} = x) = x \left(\frac{2}{3}\right)^{n-m}.
\]

Thus

\[
\text{Cov}(W_{nr}, W_{mr}) = \left(\frac{2}{3}\right)^{n-m} \text{Var}(W_{nr})
\]

\[
= \left(\frac{2}{3}\right)^{n-m} \left[\left(\frac{1}{2}\right)^{m-1} - \left(\frac{4}{9}\right)^{m-1}\right].
\]
The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$\rho_{m,n} = \frac{\left(\frac{2}{3}\right)^{n-m}\sqrt{[(\frac{1}{2})^{m-1} - (\frac{1}{2})^{m-1}]} }{\sqrt{[(\frac{1}{2})^{n-1} - (\frac{1}{2})^{n-1}]} } = \frac{\sqrt{[(\frac{2}{3})^{m-1} - 1]}}{\sqrt{[(\frac{2}{3})^{n-1} - 1]}} \to 0$$

for any fixed $m$ as $n \to \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \leq m \leq n = 5$.

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**Theorem 2.4.** The joint pdf $f^*_{m,n}$ of $W_{mr}$ and $W_{nr}$, $2 \leq m \leq n$, is given by

$$f^*_{m,n}(x, y) = \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{n-1} (-\ln(1-x))^{m-2} \times (\ln(1-x) + \ln(1-y))^{n-m-1} \frac{1-y}{(1-x)},$$

$0 < x < y < 1$.

**Proof.** Let $U_1 = \prod_{j=1}^{m} V_j$ and $U_2 = \prod_{j=1}^{n-m} V_{m+j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$f_{U_1U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m-1)}(-2 \ln u_1)^{m-2} \frac{2u_2}{\Gamma(n-m)}(-2 \ln u_2)^{n-m-1}.$$ 

Let $T_1 = U_1$ and $T_2 = U_1U_2$, then the joint pdf of $T_1$ and $T_2$ is

$$f_{T_1T_2}(t_1, t_2)
= \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)}(-2 \ln t_1)^{m-2} \frac{2t_2}{t_1^2}(-2 \ln (t_2/t_1))^{n-m-1}$$

$$= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln t_1)^{m-2} \frac{t_2}{t_1} (-\ln t_2 - \ln t_1))^{n-m-1}.$$
Sustituting $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, We obtain the joint pdf of $W_{mr}$ and $W_{nr}$, $2 \leq m < n$ as

$$f^*_{m,n}(x, y) = \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln(1 - x))^{m-2} \times [-\ln(1 - x) + \ln(1 - y)]^{n-m-1} \frac{1 - y}{1 - x},$$

$0 < x < y < 1.$ □

**Theorem 2.5.** For $m \geq 2$, $p \geq 0$ and $q \geq 0$,

$$E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q + 3} E(W_{mr})^{p+q+1}$$

**Proof.**

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] = \int_0^1 \int_x^1 [(x)^p((y)^q(1 - y))]f^*_{m,m+1}(x, y)dydx$$

$$= \int_0^1 \frac{1}{\Gamma(m-1)}2^{n-1}(-\ln(1 - x))^{m-2} \frac{1}{(1 - x)}H(x)dx, (2.9)$$

where

$$H(x) = \int_x^1 (y)^q(1 - y)dy$$

$$= \int_x^1 (y)^q(1 - y)^2dy$$

$$= \left. \frac{y^{q+1}}{(q + 1)} (1 - y)^2 \right|_x^1 + \int_x^1 \frac{2y^{q+1}}{q+1} (1 - y)dy$$

$$= -\frac{x^{q+1}}{(q + 1)} + 2\int_x^1 \frac{y^{q+1}}{(q + 1)} (1 - y)dy$$

Substituting in (2.9), we obtain

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] = \int_0^1 [(x)^p(1 - y)] \frac{1}{\Gamma(m-1)}2^{m(-\ln(1 - x))^{m-2}} \frac{1}{(1 - x)}$$

$$\times [-\frac{2x^{q+1}}{(q + 1)} + 2\int_x^1 \frac{y^{q+1}}{(q + 1)} (1 - y)dy]dx$$

$$= -\frac{2}{q + 1}E((W_{mr})^{p+q+1}) + \frac{2}{q + 1} E(W_{mr})^p(W_{m+1r})^{q+1}$$
Thus
\[
E(W_{mr})^p(W_{mr}+1)^q+1
= \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{mr}+1)^q] + \frac{2}{q + 3} E(W_{mr})^p+q+1. \quad \square
\]

**Theorem 2.6.** For \( m \geq 2, n > m \geq 2, p \geq 0 \) and \( q \geq 0 \),
\[
E(W_{mr})^p(W_{mr}+1)^q+1
= \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{mr}+1)^q] + \frac{2}{q + 3} E((W_{mr})^p(W_{n-1r})^q)
\]

**Proof.**
\[
E[(W_{mr})^p(W_{mr}+1)^q] - (W_{mr})^p(W_{mr}+1)^q+1
= \int_0^1 \int_x^1 [(x)^p\{(y)^q(1 - y)\}] f^*_{m,n}(x, y) dy dx
= \int_0^1 [(x)^p \frac{1}{\Gamma(m - 1)\Gamma(n - m)} 2^{m+k-1}
\begin{align*}
&\quad (-\ln(1 - x))^{m-2} \frac{1}{(1 - x)} H(x) dx, \\
&\quad 1
\end{align*}
(2.10)
\]
where
\[
\frac{1}{\Gamma(n - m)} H(x)
= \int_x^1 \{(y)^q(1 - y)\}[-\ln(1 - x) + \ln(1 - y)]^{n-m-1}
\times (1 - y) dy
= \int_x^1 \{(y)^q(1 - y)\}[-\ln(1 - x) + \ln(1 - y)]^{n-m-1}(1 - y)^2 dy
= \frac{y^{q+1}}{(q + 1)}[-\ln(1 - x) + \ln(1 - y)]^{n-m-1}(1 - y)^2 |_{x}
= \int_x^1 \frac{y^{q+1}}{q + 1} d[-\ln(1 - x) + \ln(1 - y)]^{n-m-1}
\times (1 - y)^2 dy
\]
\[= - \int_{x}^{1} \frac{y^{q+1}}{q+1} dy \left[- \ln(1-x) + \ln(1-y)\right]^{n-m-1} \times (1-y)^{2} dy \]

\[= 2 \int_{x}^{1} \frac{y^{q+1}}{q+1} \left[- \ln(1-x) + \ln(1-y)\right]^{n-m-1} (1-y) \times \frac{y}{\theta(q+1)} (n-m-1) \left[- \ln(1-x) + \ln(1-y)\right]^{n-m-2} \times (1-y) dy \]

Substituting H(x) in (2.10), we obtain

\[E[\left(W_{mr}\right)^{p}(W_{nr})^{q} - (W_{mr})^{p}(W_{nr})^{q+1}] = 2 \frac{E((W_{mr})^{p}(W_{nr})^{q+1}) - \frac{1}{q+1} E((W_{mr})^{p}(W_{n-1r})^{q+1})}{q+1} \]

On simplification, we obtain

\[E((W_{mr})^{p}(W_{nr})^{q+1}) = \frac{q+1}{q+3} E((W_{mr})^{p}(W_{nr})^{q+1}) + \frac{2}{q+3} E((W_{mr})^{p}(W_{n-1r})^{q+1}) \]

**Entropy of \( W_{nr} \).**

The entropy of \( W_{nr} \) is given in the following theorem.

**Theorem 2.7.** The entropy, \( I_{n} \) of \( W_{nr} \), \( n \geq 2 \), is given by

\[I_{n} = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2) \Psi(n-1) - \ln 2,\]

where \( \Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)} \).

**Proof.**

\[I_{n} = E(- \ln f_{W_{nr}}) = \int_{0}^{1} \left[ \ln \Gamma(n-1) - (n-1) \ln 2 - \ln(1-u) - (n-2) \ln(-\ln(1-u)) \times \frac{2^{n-1}(1-u)}{\Gamma(n-1)} \right]^{n-2} du \]

\[= \ln \Gamma(n-1) - (n-1) \ln 2 - H_{1} - H_{2}, \quad (2.11)\]
where

\[ H_1 = \int_0^1 \ln(1-u) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} = -\frac{n-1}{2}, \]

\[ H_2 = \int_0^1 (n-2) \ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} du \]

Substituting \(-\ln(1-u) = t\), we obtain

\[ H_2 = \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-2t} dt \]

\[ = \frac{n-2}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-t} dt - (n-2) \ln 2 \]

\[ = (n-2)[\Psi(n-1) - \ln 2]. \]

Substituting \(H_1\) and \(H_2\) in (2.11), we obtain

\[ I_n = \ln \Gamma(n-1) - (n-1) \ln 2 + \frac{n-1}{2} - (n-2)[\Psi(n-1) - \ln 2]. \]

The following table gives \(-I_n\) for \(n = 3\) to 10.

<table>
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<th>(n)</th>
<th>3</th>
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<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-I_n)</td>
<td>0.1159</td>
<td>0.3456</td>
<td>0.6698</td>
<td>1.0396</td>
<td>1.4363</td>
<td>1.8506</td>
<td>2.2775</td>
<td>2.7141</td>
</tr>
</tbody>
</table>

We will consider the estimation \(\theta\) based on the record range. When \(X_1, X_2, \ldots\) are i.i.d. with \(f(x) = \frac{1}{\theta}, \ 0 < x < \theta\).

**Theorem 2.8.** The minimum variance linear unbiased estimator of \(\theta\) is

\[ \hat{\theta} = \frac{1}{3(2^n - 1)} [3.2^{n-1}W_{n+1r} - 2^{n-2}W_{nr} - 2^{n-3}W_{n-1r} - \ldots - W_{2r}] \]

and

\[ \text{Var}(\hat{\theta}) = \frac{\theta^2}{2(2^n - 1)}. \]
Proof. Let

\[ Z_1 = d_1 W_2 r, \quad d_1 = 3.2^{1/2} \]
\[ Z_2 = d_2 (W_3 r - \frac{2}{3} W_2 r), \quad d_2 = 3.2 \]
\[ \vdots \]
\[ Z_n = d_n (W_{n+1} r - \frac{2}{3} W_n r), \quad d_n = 3.2^{2^{n-1}} \]
\[ Z' = (Z_1, Z_2, \ldots, Z_n), \]

then \( E(Z') = A\theta \), where

\[ A' = (2^{1/2}, 2, \ldots, 2^{2^n}), \quad A' A = 2(2^n - 1). \]

Then the minimum variance linear unbiased estimator (MVLUE) \( \hat{\theta} \) of \( \theta \) (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is

\[
\hat{\theta} = (A' A)^{-1} A' Z \\
= \frac{1}{2(2^n - 1)} [2^{1/2} Z_1 + 2 Z_2 + \ldots + 2^{2^n} Z_n] \\
= \frac{1}{2(2^n - 1)} [3.2^n W_{n+1} r - 2^{n-1} W_{n-1} r - 2^{n-2} W_{n-2} r - 2 W_{2r}]. \quad (2.13)
\]

\[
Var(\hat{\theta}) = \theta^2 (A' A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square
\]

For example, if \( n = 4 \), then

\[ \hat{\theta} = \frac{1}{15} [24 W_5 r - 4 W_4 r - 2 W_3 r - W_2 r] \]

and

\[
Var(\hat{\theta}) = \frac{\theta^2}{30}.
\]

Table 2.3. Coefficient of \( W_{nr} \) in MVLUE of \( \theta \).
Let $\tilde{\theta} = c\hat{\theta}$, then bias of $\tilde{\theta}$ is $(c - 1)\theta$ and mean squared error (MSE) of $\tilde{\theta}$ is $\text{MSE}(\tilde{\theta}) = c^2 \frac{\theta^2}{2(2n-1)} + (c - 1)^2 \theta^2$. The MSE of $\tilde{\theta}$ will be minimum if $c = \frac{2n+1-2}{2n+1-1}$.

The bias of $\tilde{\theta} = (c - 1)\theta = \frac{c-1}{2n+1-1}$ and $\text{MSE}(\tilde{\theta}) = \frac{1}{2n+1-1}$.

**Prediction of $W_{n+s}$**

We consider the prediction of $W_{n+s}$ based on $W_2, W_3, \ldots, W_n$.

**Theorem 2.5.** The best linear least squares predictor, $W^*_{n+s}$ of $W_{n+s}$ based on $W_2, W_3, \ldots, W_n$ is $\theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_n$.

**Proof.** The best linear least squares predictor, $W^*_{n+s}$ of $W_{n+s}$ based on $W_2, W_3, \ldots, W_n$ is

$$W^*_{n+s} = E(W_{n+s} | W_2 = x_2, W_3 = x_3, \ldots, W_n = x_n)$$

$$= E(W_{n+s} | W_n = x_n), \text{by Markov property of } W_2, W_3, \ldots$$

$$= \theta[1 - (\frac{2}{3})^s] + x_n(\frac{2}{3})^s.$$