Record Range of Uniform Distribution

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Abstract. We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function $F(x)$ and probability density function (pdf) $f(x)$. Let $R_{nl}$ be the largest observation after observing $n$th record and $R_{(ns)}$ be the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range. We will consider some distributional properties of $W_{nr}$ when $f(x) = 1$, $0 \leq x \leq 1$.

1 Introduction

Let $\{X_i, i = 1, 2, ..\}$ be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebegue measure) distribution function $F(x)$ with pdf $f(x)$. Let $R_{U(1)} = X_1, R_{U(2)}, \ldots$ be the upper records and $R_{L(1)}, R_{L(2)}, \ldots$ be lower records of $\{X_i, i = 1, 2, \ldots\}$. For various properties of record values see Ahsanullah (2005) Arnold et. al. (1998).

Suppose $R_{nl}$ is the largest observation after observing $n$th record and $R_{(ns)}$ is the smallest observation after observing the $n$th record. Then we say $W_{nr} = R_{nl} - R_{(ns)}$, $n > 1$, as the $n$th record range. The

Key words and phrases: Minimum variance linear unbiased estimation, moments, record range, uniform distribution.
joint pdf of \( f_{(nl,(ns))} \) of \( R_{nl} \) and \( R_{(ns)} \) is given by (see Arnold et al. 1998, p. 275) as

\[
f_{(nl,(ns))}(x, y) = \frac{2^{n-1}}{(n-2)!} [-\ln(F(y) + F(x))]^{n-2}f(x)f(y), \quad (1.1)
\]

\(-\infty < x < y < \infty\)

The pdf of \( f_{W_{nr}} \) of \( W_{nr} \) is given by

\[
f_{W_{nr}}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} [-\ln(F(w+u)+F(u))]^{n-2}f(w+u)f(u) \, du
\]

Suppose \( X'_i \)'s are distributed as uniform with

\[
f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)
\]

Using (1.3) in (1.2), we obtain

\[
f_{W_{nr}}(w) = \begin{cases} \frac{2n-1-\ln(1-w)}{\Gamma(n-1)} \Gamma(\ln(1-w))^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases} \quad (1.4)
\]

Figure 1.1 gives the pdf of \( W_{nr} \) for \( n = 10 \) when \( X'_i \)'s are distributed as uniform.

In this paper we will consider distributional properties of \( W_{nr} \) for the case \( X'_i \)'s are distributed as uniform distribution.

2 Main Results

Lemma 2.1. For \( n \geq 2 \) and \( 0 < x < 1 \),

\[
F_{W_{nr}}(x) = \Gamma_{-\ln(1-x)}(n-1),
\]

where

\[
\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} \, du
\]

Proof.

\[
F_{W_{nr}}(x) = \int_0^x \frac{2n-1-\ln(1-u)}{\Gamma(n-1)} \Gamma(-\ln(1-u))^{n-2} \, du
\]

\[
= \int_0^{-\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t\ln(n-2)} \, dt
\]

\[
= \Gamma_{-\ln(1-x)}(n-1). \quad \square
\]
Figure 1.1: \( f(x) = \frac{2^9(1-x)}{\Gamma(9)} (-\ln(1-x))^8 \)

Remark 2.1.

\[
1 - F_{W_{n,r}}(x) = (1 - x)^2 \sum_{j=0}^{n-2} \frac{(-2\ln(1 - x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2
\]

Lemma 2.2.

\[
F_{W_{n,r}}(x) - F_{W_{n+1,r}}(x) = \frac{f_{W_{n+1,r}}(x)}{2(1-x)}
\]

Proof.

\[
F_{W_{n,r}}(x) - F_{W_{n+1,r}}(x) = \frac{\Gamma(-2\ln(1-x))(n-1) - \Gamma(-2\ln(1-x))(n)}{\Gamma(n)} (-2\ln(1-x))^{n-1} e^{-2\ln(1-x)}
\]

\[
= \frac{(1 - x)^2}{\Gamma(n)} (-2\ln(1 - x))^{n-1}
\]

\[
= f_{W_{n+1,r}}(x)
\]

\[
= \frac{f_{W_{n+1,r}}(x)}{2(1-x)}.
\]
\begin{align*}
\mu_{nr}^p &= E(W_{nr})^p = \frac{2^{n-1}}{(n-2)!} \int_0^1 w^p (1-w) \left[-\ln(1-w)\right]^{n-2} dw \\
&= \frac{2^{n-1}(n-2)!}{(n-2)!} \int_0^1 (1-w)^p w \left[-\ln(w)\right]^{n-2} dw \\
&= 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(2+k)^{n-1}} \tag{2.1}
\end{align*}

Using \( p = 1 \) and \( p = 2 \), we can get the mean and variance of \( W_{nr} \) as

\[ E(W_{nr}) = 1 - \left(\frac{2}{3}\right)^{n-1} \]

and

\[ Var(W_{nr}) = \left(\frac{1}{2}\right)^{n-1} - \left(\frac{4}{9}\right)^{n-1}. \]

\textbf{Theorem 2.1.} Let \( \mu_n^r = E(W_{nr}^r) \), then for \( n \geq 2 \) and \( r = 1, 2, \ldots \)

\[ (r+2)\mu_n^r - r\mu_{n-1}^r = 2\mu_{n-1}^r. \tag{2.2} \]

\textbf{Proof.} \[ r(\mu_n^{r-1} - \mu_n^r) \\
= \frac{2^{n-1}}{\Gamma(n-1) 2^2} \int_0^1 [(1-w)^2 r w^{r-1} \left[-\ln(1-w)\right]^{n-2} dw \\
= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r [2(1-w)\left[-\ln(1-w)\right]^{n-2} dw \\
- \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1-w)^2 w^r \left[-\ln(1-w)\right]^{n-3} \frac{1}{1-w} dw \\
= 2\mu_n^r - 2\mu_{n-1}^r. \]

On simplification we get the result. \( \square \)

\textbf{Theorem 2.2.} For \( n \geq 2, p > 0 \),

\[ \mu_{nr}^p - \mu_{nr}^{p+1} = 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \]
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Proof.

\[ \mu_{nr}^p - \mu_{nr}^{p+1} = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1 - w) \left[ -\ln(1 - w) \right]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p (1 - w)^2 \left[ -\ln(1 - w) \right]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1 - w)^p w^2 \left[ -\ln(w) \right]^{n-2} dw \]

\[ = 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \]  \hspace{1cm} (2.3)

Theorem 2.3. For \( n \geq 1 \),

\[ 1 - W_{n+1} \overset{d}{=} \Pi_{j=1}^n V_j, \]  \hspace{1cm} (2.4)

where \( V_1, V_2, ..., V_{n-1} \) are i.i.d. with \( F(v) = v^2, \ 0 < v < 1 \).

Proof. We will show first that

\[ 1 - W_{n+1} \overset{d}{=} (1 - W_n)V_n, \]

where \( V_n \) is independent of \( 1 - W_n \) and is distributed with pdf as \( f_V(v) = 2v, \ 0 < v < 1 \).

Let \( Y_{n-1} = (1 - W_n)V_n, \ n \geq 2, \) and \( f_n \) be the pdf of \( 1 - W_n \), then

\[ f_n = \frac{2^{n-1}w}{\Gamma(n-1)} \left[ -\ln(1 - w) \right]^{n-2}, \ 0 < w < 1, \]

\[ F(y) = P(Y_{n+1} \leq y) = P(1 - W_n)V_n \leq y \]

\[ = y^2 + \int_y^1 F_n(t) \frac{2}{t^2} dt, \text{ where } F_{n-1} \text{ is the df of } Y_n \]

\[ = y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^2} dt \]

\[ = y^2 + y^2 \left[ F_n(t) \frac{1}{t^2} \right]_y^1 + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]  \hspace{1cm} (2.5)
Differentiating both sides of (2.5), with respect to $y$, we obtain

$$f(y) = f_n(y) - f_n(y) + 2y \int_y^1 f_n(t) \frac{1}{t^2} dt$$

$$= 2y \int_y^1 f_n(t) \frac{1}{t^2} dt$$

i.e.

$$\frac{f(y)}{y} = 2 \int_y^1 f_n(t) \frac{1}{t^2} dt$$

$$= 2 \int_y^1 \frac{2^{n-1}t}{\Gamma(n-1)} [-\ln(1-t)^{n-2} \frac{1}{t^2}] dt$$

$$= \frac{2n}{\Gamma(n)} [-\ln y]^{n-1}$$

Hence

$$f(y) = \frac{2n}{\Gamma(n-1)} y [-\ln y]^{n-1}$$

which is the pdf of $Y = 1 - W_{n+1r}$. Note that the sequence $Y_2, Y_3, ...$ forms a Markov chain. □

Using (2.6), we have the following representation of $W_{nr}$ for

$$1 - W_{n+1r} \overset{d}{=} \prod_{j=1}^n V_j, \ n \geq 1 \quad (2.8)$$

where $V_1, V_2, ..., V_{n-1}$ are i.i.d. with $F(v) = v^2, \ 0 < v < 1$.

The conditional expectation of

$$1 - W_{nr} | 1 - W_{mr} = x, 2 \leq m < n - 1, \ 0 < x < 1,$$

is

$$E(1 - W_{nr} | 1 - W_{mr} = x) = x \left(\frac{2}{3}\right)^{n-m}.$$ 

Thus

$$Cov(W_{nr}, W_{mr}) = \left(\frac{2}{3}\right)^{n-m} Var(W_{mr})$$

$$= \left(\frac{2}{3}\right)^{n-m} \left[\left(\frac{1}{2}\right)^{m-1} - \left(\frac{4}{9}\right)^{m-1}\right]$$.
The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$
\rho_{m,n} = \frac{(\frac{2}{3})^{n-m}\sqrt{[(\frac{1}{2})^{m-1} - (\frac{2}{3})^{m-1}]}}{\sqrt{[(\frac{1}{2})^{n-1} - (\frac{2}{3})^{n-1}]}} = \sqrt{\frac{(\frac{2}{3})^{m-1} - 1}{\sqrt{[(\frac{1}{2})^{n-1} - 1]}}}
$$

for any fixed $m$ as $n \to \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \leq m \leq n = 5$.

<table>
<thead>
<tr>
<th>m</th>
<th>2</th>
<th>3</th>
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<th>5</th>
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<tr>
<td>n</td>
<td>2</td>
<td>\frac{1}{18}</td>
<td>\frac{1}{27}</td>
<td>\frac{1}{81}</td>
</tr>
<tr>
<td>3</td>
<td>\frac{1}{27}</td>
<td>\frac{1}{324}</td>
<td>\frac{1}{486}</td>
<td>\frac{1}{729}</td>
</tr>
<tr>
<td>4</td>
<td>\frac{2}{81}</td>
<td>\frac{17}{5882}</td>
<td>\frac{217}{8748}</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>\frac{4}{243}</td>
<td>\frac{17}{729}</td>
<td>\frac{217}{8748}</td>
<td>\frac{2465}{104976}</td>
</tr>
</tbody>
</table>

**Theorem 2.4.** The joint pdf $f^*_m,n$ of $W_{mr}$ and $W_{nr}$, $2 \leq m \leq n$, is given by

$$
f^*_m,n(x, y) = \frac{1}{\Gamma(m-1)\Gamma(n-m)}2^{n-1}(-\ln(1-x))^{m-2}(-2 \ln u_1)^{m-2}\frac{2u_2}{\Gamma(n-m)}(-2 \ln u_2)^{n-m-1}.
$$

**Proof.** Let $U_1 = \Pi_{j=1}^m V_j$ and $U_2 = \Pi_{j=1}^{n-m} V_{m+j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$
f_{U_1,U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m-1)}(-2 \ln u_1)^{m-2}\frac{2u_2}{\Gamma(n-m)}(-2 \ln u_2)^{n-m-1}.
$$

Let $T_1 = U_1$ and $T_2 = U_1 U_2$, then the joint pdf of $T_1$ and $T_2$ is

$$
f_{T_1,T_2}(t_1, t_2) = \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)}(-2 \ln t_1)^{m-2}\frac{t_2}{t_1}(-2 \ln(t_2/t_1))^{n-m-1} = \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln t_1)^{m-2}\frac{t_2}{t_1}(-\ln(t_2 - \ln t_1))^{n-m-1}
$$
Sustituting \( T_1 = 1 - W_{mr} \) and \( T_2 = 1 - W_{nr} \), We obtain the joint pdf of \( W_{mr} \) and \( W_{nr} \), \( 2 \leq m < n \) as

\[
f_{m,n}^*(x, y) = \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)}(-\ln(1 - x))^{m-2} \\
\times \left[ -\ln(1 - x) + \ln(1 - y) \right]^{n-m-1} \frac{1 - y}{(1 - x)},
\]

\( 0 < x < y < 1 \). □

**Theorem 2.5.** For \( m \geq 2 \), \( p \geq 0 \) and \( q \geq 0 \),

\[
E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q + 1}{q + 3} E(W_{mr})^p(W_{m+1r})^q + \frac{2}{q + 3} E(W_{mr})^p(W_{m+1r})^{q+1}
\]

**Proof.**

\[
E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}]
= \int_0^1 \int_x^1 [(x)^p(y)^q(1 - y)] f_{m,m+1}^*(x, y) dy dx
= \int_0^1 [(x)^p \frac{1}{\Gamma(m-1)} 2^{n-1}(-\ln(1 - x))^{m-2} \frac{1}{(1 - x)} H(x) dx, (2.9)
\]

where

\[
H(x) = \int_x^1 (y)^q(1 - y)(1 - y) dy
= \int_x^1 (y)^q(1 - y)^2 dy
= \frac{y^{q+1}}{(q + 1)}(1 - y)^2 \bigg|_x^{1} - \int_x^1 \frac{2y^{q+1}}{(q + 1)}(1 - y) dx
= -\frac{x^{q+1}}{(q + 1)} + 2 \int_x^1 \frac{y^{q+1}}{(q + 1)}(1 - y) dy
\]

Substituting in (2.9), we obtain

\[
E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}]
= \int_0^1 [(x)^p(1 - y) \frac{1}{\Gamma(m-1)} 2^{n-1}(-\ln(1 - x))^{m-2} \frac{1}{(1 - x)}]
\cdot\left[ -\frac{2x^{q+1}}{(q + 1)} + 2 \int_x^1 \frac{y^{q+1}}{(q + 1)}(1 - y) dy dx
= -\frac{2}{q + 1} E((W_{mr})^{p+q+1}) + \frac{2}{q + 1} E(W_{mr})^p(W_{m+1r})^{q+1}
\]
Thus
\[ E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q + 3} E(W_{mr})^{p+q+1}. \]

**Theorem 2.6.** For \( m \geq 2, n > m \geq 2, p \geq 0 \) and \( q \geq 0 \),

\[ E(W_{mr})^p(W_{nr})^{q+1} = \frac{q + 1}{q + 3} E[(W_{mr})^p(W_{nr})^q] + \frac{2}{q + 3} E((W_{mr})^p(W_{n-1r})^q) \]

**Proof.**

\[ E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}] = \int_0^1 \int_x^1 [x]^p \{(y)^q(1-y)\} f_{m,n}(x, y) dy dx \]

\[ = \int_0^1 \{ [x]^p \frac{1}{\Gamma(m-1)\Gamma(n-m)} (\frac{1}{1-x})^{m+k-1} \} \frac{1}{(1-x)} H(x) dx, \quad (2.10) \]

where

\[ \frac{1}{\Gamma(n-m)} H(x) \]

\[ = \int_x^1 [(y)^q(1-y)][-\ln(1-x) + \ln(1-y)]^{n-m-1} \times (1-y) dy \]

\[ = \int_x^1 \frac{y^q}{q}[\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2 dy \]

\[ = \frac{y^{q+1}}{(q + 1)}[-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y)^2 \]

\[ - \int_x^1 \frac{y^{q+1}}{q + 1} \frac{d}{dy}[-\ln(1-x) + \ln(1-y)]^{n-m-1} \times (1-y)^2 dy \]
\[\begin{align*}
&= - \int_x^1 \frac{y^{q+1}}{q+1} \frac{d}{dy} \left[- \ln(1-x) + \ln(1-y)\right]^{n-m-1} \\
&\times (1-y)^2 \, dy \\
&= 2 \int_x^1 \frac{y^{q+1}}{q+1} \left[- \ln(1-x) + \ln(1-y)\right]^{n-m-1} (1-y) \\
&+ \int_x^1 \Theta(q+1) (n-m-1) \left[- \ln(1-x) + \ln(1-y)\right]^{n-m-2} \\
&\times (1-y) \, dy
\end{align*}\]

Substituting \(H(x)\) in (2.10), we obtain

\[E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{nr})^{q+1}] = \frac{2}{q+1} E((W_{mr})^p(W_{nr})^{q+1}) - \frac{1}{q+1} E((W_{mr})^p(W_{n-1r})^{q+1})\]

On simplification, we obtain

\[E((W_{mr})^p(W_{nr})^{q+1}) = \frac{q+1}{q+3} E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3} E((W_{mr})^p(W_{n-1r})^{q+1})\]

\(\square\)

**Entropy of \(W_{nr}\)**

The entropy of \(W_{nr}\) is given in the following theorem.

**Theorem 2.7.** The entropy, \(I_n\) of \(W_{nr}\), \(n \geq 2\), is given by

\[I_n = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2) \Psi(n-1) - \ln 2,\]

where \(\Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)}\).

**Proof.**

\[\begin{align*}
I_n &= E(- \ln f_{W_{nr}}) \\
&= \int_0^1 \left[\ln \Gamma(n-1) - (n-1) \ln 2 - \ln(1-u) - (n-2) \right. \\
&\quad \left. \ln(-\ln(1-u))\right]^{2n-1}(1-u)^{\frac{2n-1(1-u)}{\Gamma(n-1)}} \left[- \ln(1-u)\right]^{n-2} \, du \\
&= \ln \Gamma(n-1) - (n-1) \ln 2 - H_1 - H_2, \quad (2.11)
\end{align*}\]
where

\[ H_1 = \int_0^1 \ln(1-u) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} = -\frac{n-1}{2}, \]

\[ H_2 = \int_0^1 (n-2)\ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} du. \]

Substituting \(-\ln(1-u) = t\), we obtain

\[ H_2 = \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-2t} dt \]
\[ = \frac{n-2}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-t} dt - (n-2)\ln 2 \]
\[ = (n-2)[\Psi(n-1) - \ln 2]. \]

Substituting \(H_1\) and \(H_2\) in (2.11), we obtain

\[ I_n = \ln \Gamma(n-1) - (n-1)\ln 2 + \frac{n-1}{2} \]
\[ - (n-2)[\Psi(n-1) - \ln 2]. \]
\[ = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2. \quad \square (2.12) \]

The following table gives \(-I_n\) for \(n = 3\) to 10.

<table>
<thead>
<tr>
<th>(n)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-I_n)</td>
<td>0.1159</td>
<td>0.3456</td>
<td>0.6698</td>
<td>1.0396</td>
<td>1.4363</td>
<td>1.8506</td>
<td>2.2775</td>
<td>2.7141</td>
</tr>
</tbody>
</table>

We will consider the estimation \(\theta\) based on the record range. when \(X_1, X_2, \ldots\) are i.i.d with \(f(x) = \frac{1}{\theta}, \; 0 < x < \theta\).

**Theorem 2.8.** The minimum varaiance linear unbiased estimator of \(\hat{\theta}\) of \(\theta\) is

\[ \hat{\theta} = \frac{1}{3(2^n - 1)} [3.2^{n-1}W_{n+1r} - 2^{n-2}W_{nr} - 2^{n-3}W_{n-1r} - \cdots - W_{2r}] \]

and

\[ Var(\hat{\theta}) = \frac{\theta^2}{2(2^n - 1)}. \]
Proof. Let
\[ Z_1 = d_1 W_{2r}, d_1 = 3.2^{\frac{1}{2}} \]
\[ Z_2 = d_2 (W_{3r} - \frac{2}{3} W_{2r}), d_2 = 3.2 \]
\[ \vdots \]
\[ Z_n = d_n (W_{n+1r} - \frac{2}{3} W_{nr}), d_n = 3.2^{\frac{n}{2}} \]
\[ Z' = (Z_1, Z_2, ..., Z_n) \]
then \( E(Z') = A\theta \), where
\[ A' = (2^{\frac{1}{2}}, 2, ..., 2^{\frac{n}{2}}) \]
\[ A'A = 2(2^n - 1) \]

Then the minimum variance linear unbiased estimator (MVLUE) \( \hat{\theta} \) of \( \theta \) (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is
\[ \hat{\theta} = (A'A)^{-1} A'Z \]
\[ = \frac{1}{2(2^n - 1)} [2^{\frac{1}{2}} Z_1 + 2 Z_2 + ... + 2^{\frac{n}{2}} Z_n] \]
\[ = \frac{1}{2(2^n - 1)} [3.2^n W_{n+1r} - 2^{n-1} W_{nr} - 2^{n-2} W_{n-1r} - 2 W_{2r}] \quad (2.13) \]

\[ Var(\hat{\theta}) = \theta^2 (A'A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square \]

For example, if \( n = 4 \), then
\[ \hat{\theta} = \frac{1}{15} [24 W_{5r} - 4 W_{4r} - 2 W_{3r} - W_{2r}] \]

and
\[ Var(\hat{\theta}) = \frac{\theta^2}{30} \]

Table 2.3. Coefficient of \( W_{nr} \) in MVLUE of \( \theta \).
Let \( \hat{\theta} = c\theta \), then bias of \( \hat{\theta} \) is \( (c - 1)\theta \) and mean squared error (MSE) of \( \hat{\theta} \) is \( \text{MSE}(\hat{\theta}) = c^2 \frac{\theta^2}{2^{2n+1}} + (c - 1)^2 \theta^2 \). The MSE of \( \tilde{\theta} \) will be minimum if \( c = \frac{2^n+1-2}{2^n+1-1} \).

The bias of \( \tilde{\theta} = (c - 1)\theta = \frac{-1}{2^n+1} \) and \( \text{MSE}(\tilde{\theta}) = \frac{1}{2^n+1} \).

**Prediction of \( W_{n+s} \).**

We consider the prediction of \( W_{n+s} \) based on \( W_{2r}, W_{3r}, \ldots, W_{nr} \).

**Theorem 2.5.** The best linear least squares predictor, \( W_{n+s}^* \) of \( W_{n+s} \) based on \( W_{2r}, W_{3r}, \ldots, W_{nr} \) is \( \theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr} \).

**Proof.** The best linear least squares predictor, \( W_{n+s}^* \) of \( W_{n+s} \) based on \( W_{2r}, W_{3r}, \ldots, W_{nr} \) is

\[
W_{n+s}^* = E(W_{n+s}|W_{2r} = x_2, W_{3r} = x_3, \ldots, W_{nr} = x_n)
\]

\[= E(W_{n+s}|W_{nr} = x_n), \text{by Markov property of } W_{2r}, W_{3r}, \ldots \]

\[= \theta[1 - (\frac{2}{3})^s] + x_n (\frac{2}{3})^s.\]

**References**

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