

## Record Range of Uniform Distribution

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**Abstract.** We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function  $F(x)$  and probability density function (pdf)  $f(x)$ . Let  $R_{nl}$  be the largest observation after observing  $n$ th record and  $R_{(ns)}$  be the smallest observation after observing the  $n$ th record. Then we say  $W_{nr} = R_{nl} - R_{(ns)}$ ,  $n > 1$ , as the  $n$ th record range. We will consider some distributional properties of  $W_{nr}$  when  $f(x) = 1, 0 \leq x \leq 1$ .

### 1 Introduction

Let  $\{X_i, i = 1, 2, \dots\}$  be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebesgue measure) distribution function  $F(x)$  with pdf  $f(x)$ . Let  $R_{U(1)} = X_1, R_{U(2)}, \dots$ , be the upper records and  $R_{L(1)}, R_{L(2)}, \dots$  be lower records of  $\{X_i, i = 1, 2, \dots\}$ . For various properties of record values see Ahsanullah (2005) Arnold et. al. (1998).

Suppose  $R_{nl}$  is the largest observation after observing  $n$ th record and  $R_{(ns)}$  is the smallest observation after observing the  $n$ th record. Then we say  $W_{nr} = R_{nl} - R_{(ns)}$ ,  $n > 1$ , as the  $n$ th record range. The

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joint pdf of  $f_{(nl,(ns))}$  of  $R_{nl}$  and  $R_{(ns)}$  is given by ( see Arnold et. al. 1998, p. 275) as

$$f_{(nl,(ns))}(x, y) = \frac{2^{n-1}}{(n-2)!} [-\ln(\bar{F}(y) + F(x))]^{n-2} f(x) f(y), \quad (1.1)$$

$$-\infty < x < y < \infty$$

The pdf of  $f_{W_{nr}}$  of  $W_{nr}$  is given by

$$f_{W_{nr}}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} [-\ln(\bar{F}(w+u) + F(u))]^{n-2} f(w+u) f(u) du \quad (1.2)$$

Suppose  $X'_i$ s are distributed as uniform with

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1.3)$$

Using (1.3) in (1.2), we obtain

$$f_{W_{nr}}(w) = \begin{cases} \frac{2^{n-1}(1-w)}{\Gamma(n-1)} [-\ln\{1-w\}]^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases} \quad (1.4)$$

Figure 1.1 gives the pdf of  $W_{nr}$  for  $n = 10$  when  $X'_i$  are distributed as uniform.

In this paper we will consider distributional properties of  $W_{nr}$  for the case  $X'_i$ s are distributed as uniform distribution.

## 2 Main Results

**Lemma 2.1.** For  $n \geq 2$  and  $0 < x < 1$ ,

$$F_{W_{nr}}(x) = \Gamma_{-2\ln(1-x)}(n-1),$$

where

$$\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} du$$

**Proof.**

$$\begin{aligned} F_{W_{nr}}(x) &= \int_0^x \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} du \\ &= \int_0^{-2\ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} dt \\ &= \Gamma_{-2\ln(1-x)}(n-1). \quad \square \end{aligned}$$

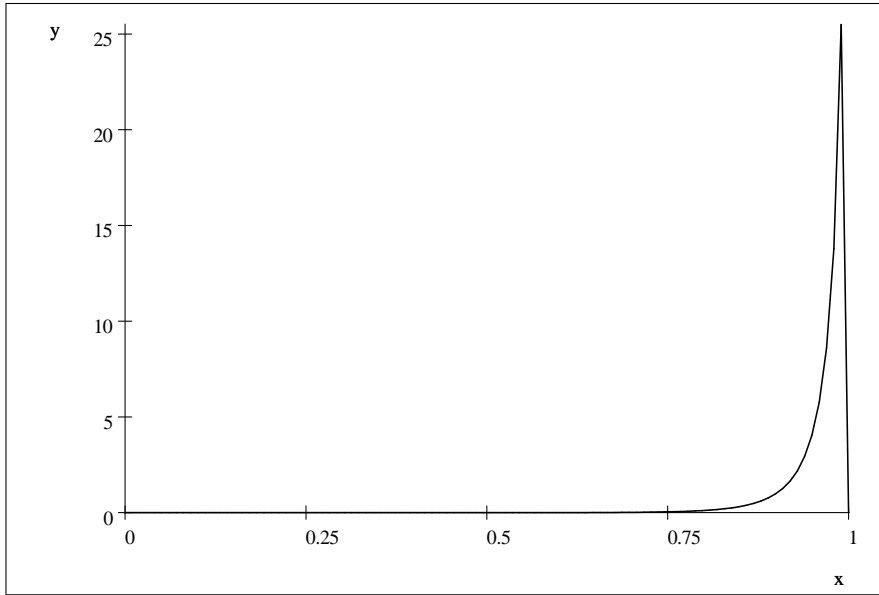


Figure 1.1:  $f(x) = \frac{2^9(1-x)}{\Gamma(9)}(-\ln(1-x))^8$

**Remark 2.1.**

$$1 - F_{W_{nr}}(x) = (1-x)^2 \sum_{j=0}^{n-2} \frac{(-2\ln(1-x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2$$

**Lemma 2.2.**

$$F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1-x)}$$

**Proof.**

$$\begin{aligned} F_{W_{nr}}(x) - F_{W_{n+1r}}(x) &= \Gamma_{-2\ln(1-x)}(n-1) - \Gamma_{-2\ln(1-x)}(n) \\ &= \frac{1}{\Gamma(n)} (-2\ln(1-x))^{n-1} e^{-2\ln(1-x)} \\ &= \frac{(1-x)^2}{\Gamma(n)} (-2\ln(1-x))^{n-1} \\ &= \frac{f_{W_{n+1r}}(x)}{2(1-x)}. \end{aligned}$$

$$\begin{aligned}
 \mu_{nr}^p &= E(W_{nr})^p = \frac{2^{n-1}}{(n-2)!} \int_0^1 w^p(1-w) [-\ln(1-w)]^{n-2} dw \\
 &= \frac{2^{n-1}(n-2)!}{(n-2)!} \int_0^1 (1-w)^p w [-\ln(w)]^{n-2} dw \\
 &= 2^{n-1} \left( \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(2+k)^{n-1}} \right) \tag{2.1}
 \end{aligned}$$

Using  $p = 1$  and  $p = 2$ , we can get the mean and variance of  $W_{nr}$  as

$$E(W_{nr}) = 1 - \left(\frac{2}{3}\right)^{n-1}$$

and

$$Var(W_{nr}) = \left(\frac{1}{2}\right)^{n-1} - \left(\frac{4}{9}\right)^{n-1}. \quad \square$$

**Theorem 2.1.** *Let  $\mu_n^r = E(W_{nr}^r)$ , then for  $n \geq 2$  and  $r = 1, 2, \dots$*

$$(r+2)\mu_n^r - r\mu_n^{r-1} = 2\mu_{n-1}^r. \tag{2.2}$$

**Proof.**

$$\begin{aligned}
 &r(\mu_n^{r-1} - \mu_n^r) \\
 &= \frac{2^{n-1}}{\Gamma(n-1)\theta^2} \int_0^1 [(1-w)^2 r w^{r-1} [-\ln(1-w)]^{n-2} dw \\
 &= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r [2(1-w) [-\ln(1-w)]^{n-2} dw \\
 &\quad - \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1-w)^2 w^r [-\ln(1-w)]^{n-3} \frac{1}{1-w} dw \\
 &= 2\mu_n^r - 2\mu_{n-1}^r.
 \end{aligned}$$

On simplification we get the result.  $\square$

**Theorem 2.2.** *For  $n \geq 2, p > 0$ ,*

$$\mu_{nr}^p - \mu_{nr}^{p+1} = 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}}$$

**Proof.**

$$\begin{aligned}
 \mu_{nr}^p - \mu_{nr}^{p+1} &= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1-w) [-\ln(1-w)]^{n-2} dw \\
 &= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p(1-w)^2 [-\ln(1-w)]^{n-2} dw \\
 &= \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1-w)^p w^2 [-\ln(w)]^{n-2} dw \\
 &= 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \tag{2.3}
 \end{aligned}$$

**Theorem 2.3.** For  $n \geq 1$ ,

$$1 - W_{n+1r} \stackrel{d}{=} \prod_{j=1}^n V_j, \tag{2.4}$$

where  $V_1, V_2, \dots, V_{n-1}$  are i.i.d. with  $F(v) = v^2, 0 < v < 1$ .

**Proof.** We will show first that

$$1 - W_{n+1r} \stackrel{d}{=} (1 - W_{nr})V_n,$$

where  $V_n$  is independent of  $1 - W_{nr}$  and is distributed with pdf as  $f_V(v) = 2v, 0 < v < 1$ .

Let  $Y_{n-1} = (1 - W_{nr})V_n, n \geq 2$ , and  $f_n$  be the pdf of  $1 - W_{nr}$ , then

$$f_n = \frac{2^{n-1}w}{\Gamma(n-1)} [-\ln(1-w)]^{n-2}, \quad 0 < w < 1,$$

$$\begin{aligned}
 F(y) &= P(Y_{n+1} \leq y) = P(1 - W_{nr})V_n \leq y) \\
 &= y^2 + \int_y^1 F_n\left(\frac{y}{v}\right) 2v dv, \text{ where } F_{n-1} \text{ is the df of } Y_{(n)} \\
 &= y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^3} dt \\
 &= y^2 + y^2 [F_n(t) \frac{-1}{t^2}]_y^1 + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
 &= y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\
 &= F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \tag{2.5}
 \end{aligned}$$

Differentiating both sides of (2.5), with respect to  $y$ , we obtain

$$\begin{aligned} f(y) &= f_n(y) - f_n(y) + 2y \int_y^1 f_n(t) \frac{1}{t^2} dt \\ &= 2y \int_y^1 f_n(t) \frac{1}{t^2} dt \end{aligned}$$

i.e.

$$\begin{aligned} \frac{f(y)}{y} &= 2 \int_y^1 f_n(t) \frac{1}{t^2} dt \\ &= 2 \int_y^1 \frac{2^{n-1} t}{\Gamma(n-1)} [-\ln(1-t)]^{n-2} \frac{1}{t^2} dt \\ &= \left[ \frac{2^n}{\Gamma(n-1)} [-\ln t]^{n-1} \frac{-1}{n-1} \right]_y^1 \\ &= \frac{2^n}{\Gamma(n)} [-\ln y]^{n-1} \end{aligned} \quad (2.6)$$

Hence

$$f(y) = \frac{2^n}{\Gamma(n-1)} y [-\ln y]^{n-1} \quad (2.7)$$

which is the pdf of  $Y = 1 - W_{n+1r}$ .

Note that the sequence  $Y_2, Y_3, \dots$  forms a Markov chain.  $\square$

Using (2.6), we have the following representation of  $W_{nr}$  for

$$1 - W_{n+1r} \stackrel{d}{=} \prod_{j=1}^n V_j, \quad n \geq 1 \quad (2.8)$$

where  $V_1, V_2, \dots, V_{n-1}$  are i.i.d. with  $F(v) = v^2$ ,  $0 < v < 1$ .

The conditional expectation of

$$1 - W_{nr} | 1 - W_{mr} = x, \quad 2 \leq m < n - 1, \quad 0 < x < 1,$$

is

$$E(1 - W_{nr} | 1 - W_{mr} = x) = x \left(\frac{2}{3}\right)^{n-m}.$$

Thus

$$\begin{aligned} Cov(W_{nr} W_{mr}) &= \left(\frac{2}{3}\right)^{n-m} Var(W_{mr}) \\ &= \left(\frac{2}{3}\right)^{n-m} \left[ \left(\frac{1}{2}\right)^{m-1} - \left(\frac{4}{9}\right)^{m-1} \right]. \end{aligned}$$

The correlation coefficient  $\rho_{m,n}$  between  $W_{nr}W_{mr}$  is given by

$$\rho_{m,n} = \frac{(\frac{2}{3})^{n-m} \sqrt{[(\frac{1}{2})^{m-1} - (\frac{4}{9})^{m-1}]} }{\sqrt{[(\frac{1}{2})^{n-1} - (\frac{4}{9})^{n-1}]} } = \frac{\sqrt{[(\frac{9}{8})^{m-1} - 1]} }{\sqrt{[(\frac{9}{8})^{n-1} - 1]} } \rightarrow 0$$

for any fixed  $m$  as  $n \rightarrow \infty$ .

The following table gives the variances and covariances of  $W_{nr}$  and  $W_{mr}$  for  $2 \leq m \leq n = 5$ .

Table 2.1. variances and covariances of  $W_{nr}$  and  $W_{mr}$

		m			
		2	3	4	5
n	2	$\frac{1}{18}$	$\frac{1}{27}$	$\frac{2}{81}$	$\frac{4}{243}$
	3	$\frac{1}{27}$	$\frac{17}{324}$	$\frac{17}{486}$	$\frac{17}{729}$
	4	$\frac{2}{81}$	$\frac{17}{486}$	$\frac{217}{5862}$	$\frac{217}{8748}$
	5	$\frac{4}{243}$	$\frac{17}{729}$	$\frac{217}{8748}$	$\frac{2465}{104976}$

**Theorem 2.4.** The joint pdf  $f_{m,n}^*$  of  $W_{mr}$  and  $W_{nr}$ ,  $2 \leq m \leq n$ , is given by

$$f_{m,n}^*(x, y) = \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{n-1} (-\ln(1-x))^{m-2} \times [-\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{(1-x)},$$

$$0 < x < y < 1.$$

**Proof.** Let  $U_1 = \prod_{j=1}^m V_j$  and  $U_2 = \prod_{j=1}^{n-m} V_{m+j}$ , then the joint pdf of  $U_1$  and  $U_2$  is given by

$$f_{U_1 U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m-1)} (-2 \ln u_1)^{m-2} \frac{2u_2}{\Gamma(n-m)} (-2 \ln u_2)^{n-m-1}.$$

Let  $T_1 = U_1$  and  $T_2 = U_1 U_2$ , then the joint pdf of  $T_1$  and  $T_2$  is

$$f_{T_1 T_2}(t_1, t_2) = \frac{2t_1}{\Gamma(m-1)\Gamma(n-m)} (-2 \ln t_1)^{m-2} 2 \frac{t_2}{t_1^2} (-2 \ln(\frac{t_2}{t_1}))^{n-m-1}$$

$$= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)} (-\ln t_1)^{m-2} \frac{t_2}{t_1} (-(\ln t_2 - \ln t_1))^{n-m-1}$$

Sustituting  $T_1 = 1 - W_{mr}$  and  $T_2 = 1 - W_{nr}$ , We obtain the joint pdf of  $W_{mr}$  and  $W_{nr}$ ,  $2 \leq m < n$  as

$$\begin{aligned} f_{m,n}^*(x, y) &= \frac{2^{n-1}}{\Gamma(m-1)\Gamma(n-m)} (-\ln(1-x))^{m-2} \\ &\times [-\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{(1-x)}, \\ &0 < x < y < 1. \quad \square \end{aligned}$$

**Theorem 2.5.** For  $m \geq 2$ ,  $p \geq 0$  and  $q \geq 0$ ,

$$\begin{aligned} E(W_{mr})^p (W_{m+1r})^{q+1} \\ = \frac{q+1}{q+3} E[(W_{mr})^p (W_{m+1r})^q] + \frac{2}{q+3} E(W_{mr})^{p+q+1} \end{aligned}$$

**Proof.**

$$\begin{aligned} &E[(W_{mr})^p (W_{m+1r})^q - (W_{mr})^p (W_{m+1r})^{q+1}] \\ &= \int_0^1 \int_x^1 [(x)^p \{(y)^q (1-y)\}] f_{m,m+1}^*(x, y) dy dx \\ &= \int_0^\theta [(x)^p \frac{1}{\Gamma(m-1)} 2^{n-1} (-\ln(1-x))^{m-2} \frac{1}{(1-x)} H(x) dx, \quad (2.9) \end{aligned}$$

where

$$\begin{aligned} H(x) &= \int_x^1 (y)^q (1-y) \{1-y\} dy \\ &= \int_x^1 (y)^q (1-y)^2 dy \\ &= \frac{y^{q+1}}{(q+1)} (1-y)^2 \Big|_x^1 + \int_x^1 \frac{2y^{q+1}}{\theta^{q+1}(q+1)} (1-y) \\ &= -\frac{x^{q+1}}{(q+1)} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)} (1-y) dy \end{aligned}$$

Substituting in (2.9), we obtain

$$\begin{aligned} &E[(W_{mr})^p (W_{m+1r})^q - (W_{mr})^p (W_{m+1r})^{q+1}] \\ &= \int_0^1 [(x)^p (1-y) \frac{1}{\Gamma(m-1)} 2^m (-\ln(1-x))^{m-2} \frac{1}{(1-x)} \\ &\cdot [-\frac{2x^{q+1}}{(q+1)} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)} (1-y) dy] dx \\ &= -\frac{2}{q+1} E((W_{mr})^{p+q+1}) + \frac{2}{q+1} E(W_{mr})^p (W_{m+1r})^{q+1} \end{aligned}$$



Thus

$$\begin{aligned}
 & E(W_{mr})^p(W_{m+1r})^{q+1} \\
 &= \frac{q+1}{q+3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q+3} E(W_{mr})^{p+q+1}. \quad \square
 \end{aligned}$$

**Theorem 2.6.** For  $m \geq 2, n > m \geq 2, p \geq 0$  and  $q \geq 0,$

$$\begin{aligned}
 & E(W_{mr})^p(W_{nr})^{q+1} \\
 &= \frac{q+1}{q+3} E((W_{mr})^p(W_{nr})^q) + \frac{2}{q+3} E(W_{mr})^{p+q+1} \\
 & E((W_{mr})^p(W_{nr})^{q+1}) \\
 &= \frac{q+1}{q+3} E[(W_{mr})^p(W_{nr})^{q+1}] + \frac{2}{q+3} E((W_{mr})^p(W_{n-1r})^q)
 \end{aligned}$$

**Proof.**

$$\begin{aligned}
 & E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}] \\
 &= \int_0^1 \int_x^1 [(x)^p \{(y)^q(1-y)\}] f_{m,n}^*(x,y) dy dx \\
 &= \int_0^1 [(x)^p \frac{1}{\Gamma(m-1)\Gamma(n-m)} 2^{m+k-1} \\
 & \quad (-\ln(1-x))^{m-2} \frac{1}{(1-x)} H(x) dx, \quad (2.10)
 \end{aligned}$$

where

$$\begin{aligned}
 & \frac{1}{\Gamma(n-m)} H(x) \\
 &= \int_x^1 \{(y)^q(1-y)\} [-\ln(1-x) + \ln(1-y)]^{n-m-1} \\
 & \quad \times (1-y) dy \\
 &= \int_x^1 \left(\frac{y}{\theta}\right)^q [-\ln(1-x) + \ln(1-y)]^{n-m-1} (1-y)^2 dy \\
 &= \frac{y^{q+1}}{(q+1)} [-\ln(1-x) + \ln(1-y)]^{n-m-1} (1-y)^2 \Big|_x^1 \\
 & \quad - \int_x^1 \frac{y^{q+1}}{q+1} \frac{d}{dy} [-\ln(1-x) + \ln(1-y)]^{n-m-1} \\
 & \quad \times (1-y)^2 dy
 \end{aligned}$$

$$\begin{aligned}
&= - \int_x^1 \frac{y^{q+1}}{q+1} \frac{d}{dy} [-\ln(1-x) + \ln(1-y)]^{n-m-1} \\
&\quad \times (1-y)^2 dy \\
&= 2 \int_x^1 \frac{y^{q+1}}{q+1} [-\ln(1-x) + \ln(1-y)]^{n-m-1} (1-y) \\
&\quad + \int_x^1 \frac{y^{q+1}}{\theta^q(q+1)} (n-m-1) [-\ln(1-x) + \ln(1-y)]^{n-m-2} \\
&\quad \times \left(1 - \frac{y}{\theta}\right) dy
\end{aligned}$$

Substituting  $H(x)$  in (2.10), we obtain

$$\begin{aligned}
&E[(W_{mr})^p (W_{nr})^q - (W_{mr})^p (W_{nr})^{q+1}] \\
&= \frac{2}{q+1} E((W_{mr})^p (W_{nr})^{q+1}) - \frac{1}{q+1} E((W_{mr})^p (W_{n-1r})^{q+1})
\end{aligned}$$

On simplification, we obtain

$$\begin{aligned}
&E((W_{mr})^p (W_{nr})^{q+1}) \\
&= \frac{q+1}{q+3} E((W_{mr})^p (W_{nr})^{q+1}) + \frac{2}{q+3} E((W_{mr})^p (W_{n-1r})^{q+1}) \quad \square
\end{aligned}$$

### Entropy of $W_{nr}$ .

The entropy of  $W_{nr}$  is given in the following theorem.

**Theorem 2.7.** *The entropy,  $I_n$  of  $W_{nr}$ ,  $n \geq 2$ , is given by*

$$I_n = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2,$$

where  $\Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)}$ .

**Proof.**

$$\begin{aligned}
I_n &= E(-\ln f_{W_{nr}}) \\
&= \int_0^1 [\ln \Gamma(n-1) - (n-1) \ln 2 - \ln(1-u) - (n-2) \\
&\quad \ln(-\ln(1-u))] \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln\{1-u\}]^{n-2} du \\
&= [\ln \Gamma(n-1) - (n-1) \ln 2 - H_1 - H_2], \quad (2.11)
\end{aligned}$$

where

$$\begin{aligned}
 H_1 &= \int_0^1 \ln(1-u) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln\{1-u\}]^{n-2} = -\frac{n-1}{2}, \\
 H_2 &= \int_0^1 (n-2)\ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln\{1-u\}]^{n-2} du
 \end{aligned}$$

Substituting  $-\ln\{1-u\} = t$ , we obtain

$$\begin{aligned}
 H_2 &= \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln te^{-2t} dt \\
 &= \frac{n-2}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln te^{-t} dt - (n-2) \ln 2 \\
 &= (n-2)[\Psi(n-1) - \ln 2].
 \end{aligned}$$

Substituting  $H_1$  and  $H_2$  in (2.11), we obtain

$$\begin{aligned}
 I_n &= \ln \Gamma(n-1) - (n-1) \ln 2 + \frac{n-1}{2} \\
 &\quad - (n-2)[\Psi(n-1) - \ln 2]. \\
 &= \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2. \quad \square(2.12)
 \end{aligned}$$

The following table gives  $-I_n$  for  $n = 3$  to 10.

Table 2.2. Values of  $-I_n$  for  $2 \leq n \leq 10$ .

$n$	3	4	5	6	7	8	9	10
$-I_n$	0.1159	0.3456	0.6698	1.0396	1,4363	1.8506	2.2775	2.7141

We will consider the estimation  $\theta$  based on the record range. when  $X_1, X_2, \dots$  are i.i.d with  $f(x) = \frac{1}{\theta}$ ,  $0 < x < \theta$ .

**Theorem 2.8.** *The minimum variance linear unbiased estimator of  $\hat{\theta}$  of  $\theta$  is*

$$\hat{\theta} = \frac{1}{3(2^n - 1)} [3 \cdot 2^{n-1} W_{n+1r} - 2^{n-2} W_{nr} - 2^{n-3} W_{n-1r} - \dots - W_{2r}]$$

and

$$Var(\hat{\theta}) = \frac{\theta^2}{2(2^n - 1)}.$$

**Proof.** Let

$$\begin{aligned} Z_1 &= d_1 W_{2r}, d_1 = 3.2^{\frac{1}{2}} \\ Z_2 &= d_2 (W_{3r} - \frac{2}{3} W_{2r}), d_2 = 3.2 \\ &\vdots \\ Z_n &= d_n (W_{n+1r} - \frac{2}{3} W_{nr}), d_n = 3.2^{\frac{n}{2}}. \\ Z' &= (Z_1, Z_2, \dots, Z_n), \end{aligned}$$

then  $E(Z') = A\theta$ , where

$$A' = (2^{\frac{1}{2}}, 2, \dots, 2^{\frac{n}{2}}). A'A = 2(2^n - 1).$$

Then the minimum variance linear unbiased estimator (MVLUE)  $\hat{\theta}$  of  $\theta$  ( see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is

$$\begin{aligned} \hat{\theta} &= (A'A)^{-1} A'Z \\ &= \frac{1}{2(2^n - 1)} [2^{\frac{1}{2}} Z_1 + 2Z_2 + \dots + 2^{\frac{n}{2}} Z_n] \\ &= \frac{1}{2(2^n - 1)} [3.2^n W_{n+1r} - 2^{n-1} W_{nr} - 2^{n-2} W_{n-1r} - 2W_{2r}]. \quad (2.13) \end{aligned}$$

$$Var(\hat{\theta}) = \theta^2 (A'A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square$$

For example, if  $n = 4$ , then

$$\hat{\theta} = \frac{1}{15} [24W_{5r} - 4W_{4r} - 2W_{3r} - W_{2r}]$$

and

$$Var(\hat{\theta}) = \frac{\theta^2}{30}.$$

Table 2.3. Coefficient of  $W_{nr}$  in MVLUE of  $\theta$ .

$n$	$W_{2r}$	$W_{3r}$	$W_{4r}$	$W_{5r}$	$W_{6r}$	$W_{7r}$	$W_{8r}$	$W_{9r}$	$W_{10r}$
2	$-\frac{1}{3}$	2							
3	$-\frac{1}{7}$	$-\frac{2}{7}$	$\frac{12}{7}$						
4	$-\frac{1}{15}$	$-\frac{2}{15}$	$-\frac{4}{15}$	$\frac{24}{15}$					
5	$-\frac{1}{31}$	$-\frac{2}{31}$	$-\frac{4}{31}$	$-\frac{8}{31}$	$\frac{48}{31}$				
6	$-\frac{1}{63}$	$-\frac{2}{63}$	$-\frac{4}{63}$	$-\frac{8}{63}$	$-\frac{16}{63}$	$\frac{96}{63}$			
7	$-\frac{1}{127}$	$-\frac{2}{127}$	$-\frac{4}{127}$	$-\frac{8}{127}$	$-\frac{16}{127}$	$-\frac{32}{127}$	$\frac{192}{127}$		
8	$-\frac{1}{255}$	$-\frac{2}{299}$	$-\frac{4}{255}$	$-\frac{8}{255}$	$-\frac{16}{255}$	$-\frac{32}{255}$	$-\frac{64}{255}$	$\frac{384}{255}$	
9	$-\frac{1}{511}$	$-\frac{2}{511}$	$-\frac{4}{511}$	$-\frac{8}{311}$	$-\frac{16}{511}$	$-\frac{32}{511}$	$-\frac{64}{511}$	$-\frac{128}{511}$	$\frac{768}{511}$

Let  $\tilde{\theta} = c\hat{\theta}$ , then bias of  $\tilde{\theta}$  is  $(c - 1)\theta$  and mean squared error (MSE) of  $\tilde{\theta}$  is  $MSE(\tilde{\theta}) = c^2 \frac{\theta^2}{2(2^n - 1)} + (c - 1)^2 \theta^2$ . The MSE of  $\tilde{\theta}$  will be minimum if  $c = \frac{2^{n+1} - 2}{2^{n+1} - 1}$ .

The bias of  $\tilde{\theta} = (c - 1)\theta = \frac{-1}{2^{n+1} - 1}$  and  $MSE(\tilde{\theta}) = \frac{1}{2^{n+1} - 1}$ .

**Prediction of  $W_{n+sr}$ .**

We consider the prediction of  $W_{n+sr}$  based on  $W_{2r}, W_{3r}, \dots, W_{nr}$ .

**Theorem 2.5.** *The best linear least squares predictor,  $W_{n+sr}^*$  of  $W_{n+sr}$  based on  $W_{2r}, W_{3r}, \dots, W_{nr}$  is  $\theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr}$ .*

**Proof.** The best linear least squares predictor,  $W_{n+sr}^*$  of  $W_{n+sr}$  based on  $W_{2r}, W_{3r}, \dots, W_{nr}$  is

$$\begin{aligned} W_{n+sr}^* &= E(W_{n+sr} | W_{2r} = x_2, W_{3r} = x_3, \dots, W_{nr} = x_n) \\ &= E(W_{n+sr} | W_{nr} = x_n), \text{ by Markov property of } W_{2r}, W_{3r}, \dots \\ &= \theta[1 - (\frac{2}{3})^s] + x_n (\frac{2}{3})^s. \end{aligned}$$

**References**

Ahsanullah, M. (2005), Record Statistics-Theory and Methods. Lanham, Maryland: University Press of America.

Ahsanullah, M. and Nevzorov, W. B. (2005), Order Statistics - Examples and Exercises. New York: Nova Science Publishers.

Arnold, B. C., Balakrishnan, N., and Nagaraja, H. N. (1998), Records. New York: Wiley and Sons.

David, H. A. and Nagaraja, H. N. (2003), *Order Statistics*. 3rd Edition. New York: John Wiley and Sons.