Record Range of Uniform Distribution

M. Ahsanullah

Department of Management Sciences, Rider University, Lawrenceville, NJ, USA.

Abstract. We consider a sequence of independent and identically distributed (iid) random variables with absolutely continuous distribution function \( F(x) \) and probability density function (pdf) \( f(x) \). Let \( R_{nl} \) be the largest observation after observing \( n \)th record and \( R_{(ns)} \) be the smallest observation after observing the \( n \)th record. Then we say \( W_{nr} = R_{nl} - R_{(ns)} \), \( n > 1 \), as the \( n \)th record range. We will consider some distributional properties of \( W_{nr} \) when \( f(x) = 1, 0 \leq x \leq 1 \).

1 Introduction

Let \( \{X_i, i = 1, 2, \ldots\} \) be a sequence of independent and identically distributed random variables with an absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with pdf \( f(x) \). Let \( R_{U(1)} = X_1, R_{U(2)}, \ldots \), be the upper records and \( R_{L(1)}, R_{L(2)}, \ldots \), be lower records of \( \{X_i, i = 1, 2, \ldots\} \). For various properties of record values see Ahsanullah (2005) Arnold et. al. (1998).

Suppose \( R_{nl} \) is the largest observation after observing \( n \)th record and \( R_{(ns)} \) is the smallest observation after observing the \( n \)th record. Then we say \( W_{nr} = R_{nl} - R_{(ns)}, n > 1 \), as the \( n \)th record range. The

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The joint pdf of $f_{(nl,(ns))}$ of $R_{nl}$ and $R_{(ns)}$ is given by (see Arnold et. al. 1998, p. 275) as

$$f_{(nl,(ns))}(x, y) = \frac{2^{n-1}}{(n-2)!} \left( \ln(F(y) + F(x)) \right)^{n-2} f(x) f(y), \quad -\infty < x < y < \infty$$

The pdf of $f_{w_{nr}}$ of $W_{nr}$ is given by

$$f_{w_{nr}}(w) = \int_{-\infty}^{\infty} \frac{2^{n-1}}{(n-2)!} \left( \ln(F(w+u) + F(u)) \right)^{n-2} f(w+u) f(u) \, du$$

Suppose $X_i's$ are distributed as uniform with

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Using (1.3) in (1.2), we obtain

$$f_{w_{nr}}(w) = \begin{cases} \frac{2^{n-1}(1-w)}{\Gamma(n-1)} \left( \ln(1-w) \right)^{n-2}, & 0 < w < 1, n \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

Figure 1.1 gives the pdf of $W_{nr}$ for $n = 10$ when $X_i's$ are distributed as uniform.

In this paper we will consider distributional properties of $W_{nr}$ for the case $X_i's$ are distributed as uniform distribution.

## 2 Main Results

**Lemma 2.1.** For $n \geq 2$ and $0 < x < 1$,

$$F_{w_{nr}}(x) = \Gamma_{-2 \ln(1-x)}(n-1),$$

where

$$\Gamma_x(r) = \int_0^x \frac{1}{\Gamma(r)} u^{r-1} e^{-u} \, du$$

**Proof.**

$$F_{w_{nr}}(x) = \int_0^x \frac{2^{n-1}(1-u)}{\Gamma(n-1)} \left( \ln(1-u) \right)^{n-2} \, du$$

$$= \int_0^{-2 \ln(1-x)} \frac{1}{\Gamma(n-1)} e^{-t} t^{n-2} \, dt$$

$$= \Gamma_{-2 \ln(1-x)}(n-1). \quad \square$$
Figure 1.1: \( f(x) = \frac{2^g(1-x)}{\Gamma(9)} (-\ln(1-x))^8 \)

**Remark 2.1.**

\[ 1 - F_{W_{nr}}(x) = (1-x)^2 \sum_{j=0}^{n-2} \frac{(-2\ln(1-x))^j}{j!}, \quad 0 < x < 1, \quad n \geq 2 \]

**Lemma 2.2.**

\[ F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \frac{f_{W_{n+1r}}(x)}{2(1-x)} \]

**Proof.**

\[
F_{W_{nr}}(x) - F_{W_{n+1r}}(x) = \Gamma_{-2\ln(1-x)}(n-1) - \Gamma_{-2\ln(1-x)}(n) \\
= \frac{1}{\Gamma(n)} (-2\ln(1-x))^{n-1} e^{-2\ln(1-x)} \\
= \frac{(1-x)^2}{\Gamma(n)} (-2\ln(1-x))^{n-1} \\
= \frac{f_{W_{n+1r}}(x)}{2(1-x)}. 
\]
\[ \mu^p_{nr} = E(W_{nr})^p = \frac{2^{n-1}}{(n-2)!} \int_0^1 w^p (1-w) \left[ -\ln(1-w) \right]^{n-2} dw \]

\[ = \frac{2^{n-1}(n-2)!}{(n-2)!} \int_0^1 (1-w)^p w \left[ -\ln(w) \right]^{n-2} dw \]

\[ = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(2+k)^{n-1}} \] (2.1)

Using \( p = 1 \) and \( p = 2 \), we can get the mean and variance of \( W_{nr} \) as

\[ E(W_{nr}) = 1 - \left( \frac{2}{3} \right)^{n-1} \]

and

\[ \text{Var}(W_{nr}) = \left( \frac{1}{2} \right)^{n-1} - \left( \frac{4}{9} \right)^{n-1}. \]

**Theorem 2.1.** Let \( \mu^r_n = E(W_{nr})^r \), then for \( n \geq 2 \) and \( r = 1, 2, \ldots \)

\[ (r+2)\mu^r_n - r\mu^{r-1}_n = 2\mu^r_{n-1}. \] (2.2)

**Proof.**

\[ r(\mu^r_{n-1} - \mu^r_n) \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)\theta^2} \int_0^1 [(1-w)^2 w^{r-1} [-\ln(1-w)]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^r [2(1-w)[-\ln(1-w)]^{n-2} dw \]

\[ - \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^1 (1-w)^2 w^r [-\ln(1-w)]^{n-3} \frac{1}{1-w} dw \]

\[ = 2\mu^r_n - 2\mu^r_{n-1}. \]

On simplification we get the result. \( \Box \)

**Theorem 2.2.** For \( n \geq 2, p > 0, \)

\[ \mu^p_{nr} - \mu^{p+1}_{nr} = 2^{n-1} \sum_{k=0}^{p} \binom{p}{k} \frac{(-1)^k}{(3+k)^{n-1}} \]
Proof.

\[ \mu_p^{nr} - \mu_{p+1}^{nr} = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (w^p - w^{p+1})(1 - w) \left[-\ln(1 - w)\right]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 w^p (1 - w)^2 \left[-\ln(1 - w)\right]^{n-2} dw \]

\[ = \frac{2^{n-1}}{\Gamma(n-1)} \int_0^1 (1 - w)^p w^2 \left[-\ln(w)\right]^{n-2} dw \]

\[ = 2^{n-1} \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k}{(3 + k)^{n-1}} \] (2.3)

**Theorem 2.3.** For \( n \geq 1 \),

\[ 1 - W_{n+1r} = \prod_{j=1}^n V_j, \] (2.4)

where \( V_1, V_2, ..., V_{n-1} \) are i.i.d. with \( F(v) = v^2, \quad 0 < v < 1 \).

**Proof.** We will show first that

\[ 1 - W_{n+1r} = (1 - W_{nr})V_n, \]

where \( V_n \), is independent of \( 1 - W_{nr} \) and is distributed with pdf as \( f_V(v) = 2v, \quad 0 < v < 1 \).

Let \( Y_{n-1} = (1 - W_{nr})V_n, \quad n \geq 2 \), and \( f_n \) be the pdf of \( 1 - W_{nr} \), then

\[ f_n = \frac{2^{n-1}w}{\Gamma(n-1)} \left[-\ln(1 - w)\right]^{n-2}, \quad 0 < w < 1, \]

\[ F(y) = P(Y_{n+1} \leq y) = P(1 - W_{nr})V_n \leq y \]

\[ = y^2 + \int_y^1 F_n(t) \frac{2}{t^2} dt, \quad \text{where } F_{n-1} \text{ is the df of } Y(n) \]

\[ = y^2 + y^2 \int_y^1 F_n(t) \frac{2}{t^2} dt \]

\[ = y^2 + y^2 \left[ F_n(t) \frac{-1}{t^2} \right]_y^1 + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = y^2 - y^3 + F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \]

\[ = F_n(y) + y^2 \int_y^1 f_n(t) \frac{1}{t^2} dt \] (2.5)
Differentiating both sides of (2.5), with respect to $y$, we obtain

$$f(y) = f_n(y) - f_n(y) + 2y \int_y^1 f_n(t) \frac{1}{t^2} dt$$

i.e.

$$\frac{f(y)}{y} = 2 \int_y^1 f_n(t) \frac{1}{t^2} dt$$

$$= 2 \int_y^1 \frac{2^{n-1} t}{\Gamma(n-1)} [-\ln(1-t)^{n-2}] \frac{1}{t^2} dt$$

$$= \frac{2^n}{\Gamma(n-1)} [-\ln t]^{n-1} - \frac{1}{n-1} y^{n-1}$$

$$= \frac{2^n}{\Gamma(n)} [-\ln y]^{n-1}$$

(2.6)

Hence

$$f(y) = \frac{2^n}{\Gamma(n-1)} y[-\ln y]^{n-1}$$

(2.7)

which is the pdf of $Y = 1 - W_{n+1r}$.

Note that the sequence $Y_2, Y_3, ...$ forms a Markov chain. □

Using (2.6), we have the following representation of $W_{nr}$ for

$$1 - W_{n+1r} \overset{d}{=} \Pi_{j=1}^n V_j, \quad n \geq 1$$

(2.8)

where $V_1, V_2, ..., V_{n-1}$ are i.i.d. with $F(v) = v^2, \quad 0 < v < 1$.

The conditional expectation of

$$1 - W_{nr} | 1 - W_{mr} = x, \quad 2 \leq m < n - 1, \quad 0 < x < 1,$$

is

$$E(1 - W_{nr} | 1 - W_{mr} = x) = x (\frac{2}{3})^{n-m}.$$

Thus

$$\text{Cov}(W_{nr}, W_{mr}) = (\frac{2}{3})^{n-m} \text{Var}(W_{mr})$$

$$= (\frac{2}{3})^{n-m} [ (\frac{1}{2})^{m-1} - (\frac{4}{9})^{m-1} ].$$
The correlation coefficient $\rho_{m,n}$ between $W_{nr}W_{mr}$ is given by

$$\rho_{m,n} = \frac{\left(\frac{2}{3}\right)^{n-m}}{\sqrt{\left[\left(\frac{2}{3}\right)^{m-1} - \left(\frac{2}{3}\right)^{n-1}\right]}} = \frac{\sqrt{\left[\left(\frac{2}{3}\right)^{m-1} - 1\right]}}{\sqrt{\left[\left(\frac{2}{3}\right)^{n-1} - 1\right]}} \to 0$$

for any fixed $m$ as $n \to \infty$.

The following table gives the variances and covariances of $W_{nr}$ and $W_{mr}$ for $2 \leq m \leq n = 5$.

<table>
<thead>
<tr>
<th>m</th>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>$\frac{1}{18}$</td>
<td>$\frac{1}{27}$</td>
<td>$\frac{2}{81}$</td>
<td>$\frac{4}{243}$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>$\frac{4}{243}$</td>
<td>$\frac{17}{729}$</td>
<td>$\frac{217}{8748}$</td>
<td>$\frac{2465}{104976}$</td>
</tr>
</tbody>
</table>

**Theorem 2.4.** The joint pdf $f_{m,n}^*$ of $W_{mr}$ and $W_{nr}$, $2 \leq m \leq n$, is given by

$$f_{m,n}^*(x, y) = \frac{1}{\Gamma(m-1)\Gamma(n-m)}2^{n-1}(-\ln(1-x))^{m-2}$$

$$\times [-\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{(1-x)},$$

$0 < x < y < 1$.

**Proof.** Let $U_1 = \prod_{j=1}^{m} V_j$ and $U_2 = \prod_{j=1}^{n-m} V_{m+j}$, then the joint pdf of $U_1$ and $U_2$ is given by

$$f_{U_1U_2}(u_1, u_2) = \frac{2u_1}{\Gamma(m-1)}(-2\ln u_1)^{m-2} \frac{2u_2}{\Gamma(n-m)}(-2\ln u_2)^{n-m-1}.$$
Sustituting $T_1 = 1 - W_{mr}$ and $T_2 = 1 - W_{nr}$, We obtain the joint pdf of $W_{mr}$ and $W_{nr}$, $2 \leq m < n$ as

$$f_{m,n}^*(x, y) = \frac{2^{n-1}}{\Gamma(m-1) \Gamma(n-m)} (-\ln(1-x))^{m-2} \times [-\ln(1-x) + \ln(1-y)]^{n-m-1} \frac{1-y}{1-x},$$

$0 < x < y < 1$. □

**Theorem 2.5.** For $m \geq 2$, $p \geq 0$ and $q \geq 0$,

$$E(W_{mr})^p(W_{m+1r})^{q+1} = \frac{q+1}{q+3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q+3} E(W_{mr})^{p+q+1}$$

**Proof.**

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] = \int_0^1 \int_x^1 [(x)^p(y)^q(1-y)] f_{m+1}^*(x, y) dy dx$$

$$= \int_0^1 \frac{1}{\Gamma(m-1)} 2^{n-1} (-\ln(1-x))^{m-2} \frac{1}{(1-x)} H(x) dx, (2.9)$$

where

$$H(x) = \int_x^1 (y)^q(1-y) dy$$

$$= \int_x^1 (y)^q(1-y)^2 dy$$

$$= \frac{y^{q+1}}{(q+1)} (1-y)^2 \bigg|_x^1 + \int_x^1 \frac{2y^{q+1}}{q+1} (1-y)$$

$$= -\frac{x^{q+1}}{(q+1)} + 2 \int_x^1 \frac{y^{q+1}}{(q+1)} (1-y) dy$$

Substituting in (2.9), we obtain

$$E[(W_{mr})^p(W_{m+1r})^q - (W_{mr})^p(W_{m+1r})^{q+1}] = \int_0^1 [(x)^p(1-y)] (1-y) \frac{1}{\Gamma(m-1)} 2^{n-1} (-\ln(1-x))^{m-2} \frac{1}{(1-x)}$$

$$- \frac{2}{q+1} E((W_{mr})^{p+q+1}) + \frac{2}{q+1} E(W_{mr})^{p+q+1}$$
Thus
\[ E(W_{mr})^p(W_{m+1r})^{q+1} \]
\[ = \frac{q+1}{q+3} E[(W_{mr})^p(W_{m+1r})^q] + \frac{2}{q+3} E(W_{mr})^{p+q+1}. \]

**Theorem 2.6.** For \( m \geq 2, n > m \geq 2, p \geq 0 \) and \( q \geq 0 \),
\[ E(W_{mr})^p(W_{nr})^q = \frac{q+1}{q+3} E[(W_{mr})^p(W_{nr})^q] + \frac{2}{q+3} E[(W_{mr})^p(W_{n-1r})^q] \]

**Proof.**
\[ E[(W_{mr})^p(W_{nr})^q - (W_{mr})^p(W_{n+1r})^{q+1}] \]
\[ = \int_{0}^{1} \int_{x}^{1} \left[ (x)^p \{(y)^q(1-y)\} \right] f_{m,n}(x,y) dy dx \]
\[ = \int_{0}^{1} \left[ (x)^p \frac{1}{\Gamma(m-1)\Gamma(n-m)} \right]^2 m+k-1 \]
\[ \left[ -\ln(1-x) + \ln(1-y) \right]^{n+m-2} \frac{1}{(1-x)} H(x) dx, \] (2.10)

where
\[ (1-y)(x) \]
\[ = \int_{x}^{1} \left[ (y)^q(1-y) \right] \left[ -\ln(1-x) + \ln(1-y) \right]^{n-m-1} \]
\[ \times (1-y) dy \]
\[ = \int_{x}^{1} \frac{y^{q+1}}{(q+1)} \left[ -\ln(1-x) + \ln(1-y) \right]^{n-m-1}(1-y)^2 dy \]
\[ = \int_{x}^{1} \frac{y^{q+1}}{(q+1)} \left[ -\ln(1-x) + \ln(1-y) \right]^{n-m-1}(1-y)^2 \]
\[ \times (1-y)^2 dy \]
\[= - \int_{x}^{1} \frac{y^{q+1}}{q+1} \frac{d}{dy}[-\ln(1-x) + \ln(1-y)]^{n-m-1} \times (1-y)^{2} \ dy \]

\[= 2 \int_{x}^{1} \frac{y^{q+1}}{q+1} [-\ln(1-x) + \ln(1-y)]^{n-m-1}(1-y) \]

\[+ \int_{x}^{1} \frac{y^{q+1}}{\theta^{(q+1)}} (n-m-1)[-\ln(1-x) + \ln(1-y)]^{n-m-2} \times (1-y)^{\frac{1}{\theta}} \ dy \]

Substituting \(H(x)\) in (2.10), we obtain

\[E[(W_{mr})^{p}(W_{nr})^{q} - (W_{mr})^{p}(W_{nr})^{q+1}] \]

\[= \frac{2}{q+1} E((W_{mr})^{p}(W_{nr})^{q+1}) - \frac{1}{q+1} E((W_{mr})^{p}(W_{nr-1})^{q+1}) \]

On simplification, we obtain

\[E((W_{mr})^{p}(W_{nr})^{q+1}) = \frac{q+1}{q+3} E((W_{mr})^{p}(W_{nr})^{q+1}) + \frac{2}{q+3} E((W_{mr})^{p}(W_{nr-1})^{q+1}) \]

\[\square \]

**Entropy of \(W_{nr}\).**

The entropy of \(W_{nr}\) is given in the following theorem.

**Theorem 2.7.** The entropy, \(I_{n}\) of \(W_{nr}\), \(n \geq 2\), is given by

\[I_{n} = \ln \Gamma(n-1) + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2, \]

where \(\Psi(n-1) = \frac{\Gamma'(n-1)}{\Gamma(n-1)}\).

**Proof.**

\[I_{n} = E(-\ln f_{W_{nr}}) \]

\[= \int_{0}^{1} [\ln \Gamma(n-1) - (n-1)\ln 2 - \ln(1-u) - (n-2) \ln(-\ln(1-u))]^{n-2} \frac{2^{n-1}(1-u)}{\Gamma(n-1)} \ dn^{2} (1-u) \]

\[= |\ln \Gamma(n-1) - (n-1)\ln 2 - H_{1} - H_{2}, \]

(2.11)
where

\[
H_1 = \int_0^1 \ln(1-u) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} (-\ln(1-u))^{n-2} = -\frac{n-1}{2},
\]

\[
H_2 = \int_0^1 (n-2) \ln(-\ln(1-u)) \frac{2^{n-1}(1-u)}{\Gamma(n-1)} [-\ln(1-u)]^{n-2} du
\]

Substituting \(-\ln(1-u) = t\), we obtain

\[
H_2 = \frac{2^{n-1}(n-2)}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-2t} dt
\]

\[
= \frac{n-2}{\Gamma(n-1)} \int_0^\infty t^{n-2} \ln t e^{-t} dt - (n-2) \ln 2
\]

\[
= (n-2)[\Psi(n-1) - \ln 2].
\]

Substituting \(H_1\) and \(H_2\) in (2.11), we obtain

\[
I_n = \ln \Gamma(n-1) - (n-1) \ln 2 + \frac{n-1}{2} - (n-2)\Psi(n-1) - \ln 2. \quad \square (2.12)
\]

The following table gives \(-I_n\) for \(n = 3\) to 10.

<table>
<thead>
<tr>
<th>(n)</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-I_n)</td>
<td>0.1159</td>
<td>0.3456</td>
<td>0.6698</td>
<td>1.0396</td>
<td>1.4363</td>
<td>1.8506</td>
<td>2.2775</td>
<td>2.7141</td>
</tr>
</tbody>
</table>

We will consider the estimation \(\theta\) based on the record range. when \(X_1, X_2, \ldots\) are i.i.d with \(f(x) = \frac{1}{\theta}, \ 0 < x < \theta\).

**Theorem 2.8.** The minimum variance linear unbiased estimator of \(\theta\) of \(\theta\) is

\[
\hat{\theta} = \frac{1}{3(2^n - 1)} [3.2^{n-1}W_{n+1r} - 2^{n-2}W_{nr} - 2^{n-3}W_{n-1r} - \ldots - W_{2r}]
\]

and

\[
Var(\hat{\theta}) = \frac{\theta^2}{2(2^n - 1)}.
\]
Proof. Let
\[ Z_1 = d_1 W_{2r}, \quad d_1 = 3.2^{\frac{1}{2}} \]
\[ Z_2 = d_2 (W_{3r} - \frac{2}{3} W_{2r}), \quad d_2 = 3.2 \]
\[ \vdots \]
\[ Z_n = d_n (W_{n+1r} - \frac{2}{3} W_{nr}), \quad d_n = 3.2^{\frac{n}{2}}. \]
\[ Z' = (Z_1, Z_2, \ldots, Z_n), \]
then \( E(Z') = A\theta \), where
\[ A' = (2^{\frac{1}{2}}, 2, \ldots, 2^{\frac{n}{2}}). A' A = 2(2^n - 1). \]
Then the minimum variance linear unbiased estimator (MVLUE) \( \hat{\theta} \) of \( \theta \) (see Ahsanullah and Nevzorov (2005), Nagaraja and David (2003)) is
\[ \hat{\theta} = (A'A)^{-1} A' Z \]
\[ = \frac{1}{2(2^n - 1)} [2^{\frac{1}{2}} Z_1 + 2 Z_2 + \ldots + 2^{\frac{n}{2}} Z_n] \]
\[ = \frac{1}{2(2^n - 1)} [3.2^n W_{n+1r} - 2^{n-1} W_{nr} - 2^{n-2} W_{n-1r} - W_{2r}]. \quad (2.13) \]
\[ Var(\hat{\theta}) = \theta^2 (A'A)^{-1} = \frac{\theta^2}{2(2^n - 1)}. \quad \square \]
For example, if \( n = 4 \), then
\[ \hat{\theta} = \frac{1}{15} [24 W_{5r} - 4 W_{4r} - 2 W_{3r} - W_{2r}] \]
and
\[ Var(\hat{\theta}) = \frac{\theta^2}{30}. \]

Table 2.3. Coefficient of \( W_{nr} \) in MVLUE of \( \theta \).
Let $\tilde{\theta} = c\hat{\theta}$, then bias of $\tilde{\theta}$ is $(c-1)\theta$ and mean squared error (MSE) of $\tilde{\theta}$ is $\text{MSE}(\tilde{\theta}) = c^2 \frac{\theta^2}{2n+1} + (c-1)^2\theta^2$. The MSE of $\tilde{\theta}$ will be minimum if $c = \frac{2n+1-2}{2n+1-1}$.

The bias of $\hat{\theta} = (c-1)\theta = \frac{-1}{2n+1-1}$ and $\text{MSE}(\hat{\theta}) = \frac{1}{2n+1-1}$.

**Prediction of $W_{n+sr}$**

We consider the prediction of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$.

**Theorem 2.5.** The best linear least squares predictor, $W^*_n$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is $\theta[1 - (\frac{2}{3})^s] + (\frac{2}{3})^s W_{nr}$.

**Proof.** The best linear least squares predictor, $W^*_n$ of $W_{n+sr}$ based on $W_{2r}, W_{3r}, \ldots, W_{nr}$ is

$$W^*_n = E(W_{n+sr}|W_{2r} = x_2, W_{3r} = x_3, \ldots, W_{nr} = x_n)$$

$$= E(W_{n+sr}|W_{nr} = x_n), \text{by Markov property of } W_{2r}, W_{3r}, \ldots$$

$$= \theta[1 - (\frac{2}{3})^s] + x_n (\frac{2}{3})^s.$$  

**References**

