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Density Estimators for Truncated Dependent Data

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Abstract. In some long term studies, a series of dependent and possibly truncated lifetime data may be observed. Suppose that the lifetimes have a common continuous distribution function F. A popular stochastic measure of the distance between the density function f of the lifetimes and its kernel estimate f_n is the integrated square error (ISE). In this paper, we derive a central limit theorem for the integrated square error of the kernel density estimators in the left-truncation model. It is assumed that the lifetime observations form a stationary strong mixing sequence. A central limit theorem (CLT) for the ISE of the kernel hazard rate estimators is also presented.

Keywords. Bandwidth, integrated square error, Kaplan-Meier estimator, kernel density estimator, strong mixing, truncated dependent data.

MSC: 62G20, 62G07.

1 Introduction

In medical follow-up or in engineering life testing studies, one may not be able to observe the variable of interest, referred to hereafter as the lifetime. Among the different forms in which incomplete data appear,

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right censoring and left-truncation are two common ones. Left truncation may occur if the time origin of the lifetime precedes the time origin of the study. Only subjects that fail after the start of the study are being followed, otherwise they are left truncated. Woodroofe (1985) reviews examples from astronomy and economy where such data may occur. In the left-truncation model, if the lifetime observations in the sample are assumed to be mutually independent, the nonparametric product-limit (PL) estimator of survival function has been studied extensively by many authors during recent years, such as Woodroofe (1985), Chao and Lo (1988), Keiding and Gill (1990), Stute (1993) and others. Let $\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_N}$ be a sequence of the lifetime variables which may not be mutually independent, but have a common unknown distribution function (d.f.) F with a density function $f = F'$. Let $\mathbf{T_1}, \mathbf{T_2}, \ldots, \mathbf{T_N}$ be a sequence of independent and identically distributed (i.i.d) random variables (rv's) with continuous d.f. G; they are also assumed to be independent of $\mathbf{X_i}$'s. In the left-truncation model, $(\mathbf{X_i}, \mathbf{T_i})$ is observed only when $\mathbf{X_i} \geq \mathbf{T_i}$. Let $(X_1, T_1), \ldots, (X_n, T_n)$ be the actually observed sample (i.e., $X_i \geq T_i$), and put $\gamma := \mathbf{P}(\mathbf{T_1} \leq \mathbf{X_1}) > 0$, where $\mathbf{P}$ is the absolute probability (related to the $\mathbf{N}$-sample). Note that n itself is a r.v. and that γ can be estimated by $n/\mathbf{N}$ (although this estimator cannot be calculated since $\mathbf{N}$ is unknown). Assume, without loss of generality, that $\mathbf{X_i}$ and $\mathbf{T_i}$ are nonnegative random variables, $i = 1, \ldots, \mathbf{N}$. For any d.f. L denote the left and right endpoints of its support by $a_L = \inf\{x : L(x) > 0\}$ and $b_L = \sup\{x : L(x) < 1\}$, respectively. Then under the current model, as discussed by Woodroofe (1985), we assume that $a_G \leq a_F$ and $b_G \leq b_F$. Define

$$\begin{aligned} C(x) &= \mathbf{P}(\mathbf{T_1} \leq x \leq \mathbf{X_1} | \mathbf{T_1} \leq \mathbf{X_1}) \\ &= \mathbb{P}(T_1 \leq x \leq X_1) = \gamma^{-1} G(x)(1 - F(x)), \end{aligned} \tag{1}$$

where $\mathbb{P}(.) = \mathbf{P}(.|n)$ is the conditional probability (related to the n-sample) and consider its empirical estimate

$$C_n(x) = n^{-1} \sum_{i=1}^{n} I(T_i \leq x \leq X_i), \tag{2}$$

where $I(.)$ is the indicator function. Then the PL estimator $\widehat{F}_n$ of F is given by

$$\widehat{F}_n(x) = 1 - \prod_{X_i \leq x} \left(1 - \frac{1}{nC_n(X_i)}\right). \tag{3}$$

The cumulative hazard function $\Lambda(x)$ is defined by

$$\Lambda(x) = \int_0^x \frac{f(u)}{1 - F(u)} du. \tag{4}$$

Let

$$F^*(x) = \mathbf{P}(\mathbf{X_1} \leq x | \mathbf{T_1} \leq \mathbf{X_1}) = \mathbb{P}(X_1 \leq x) = \gamma^{-1} \int_0^x G(u) dF(u), \tag{5}$$

be the d.f. of the observed lifetimes. Its empirical estimator is given by

$$F_n^*(x) = n^{-1} \sum_{i=1}^n I(X_i \leq x).$$

On the other hand, the d.f. of the observed T_i's is given by

$$G^*(x) = \mathbf{P}(\mathbf{T_1} \leq x | \mathbf{T_1} \leq \mathbf{X_1}) = \mathbb{P}(T_1 \leq x) = \gamma^{-1} \int_0^\infty G(x \wedge u) dF(u),$$

and is estimated by

$$G_n^*(x) = n^{-1} \sum_{i=1}^n I(T_i \leq x).$$

It then follows from (1) and (2) that

$$C(x) = G^*(x) - F^*(x), \quad C_n(x) = G_n^*(x) - F_n^*(x-). \tag{6}$$

Finally (1), (4) and (5) give

$$\Lambda(x) = \int_0^x \frac{dF^*(u)}{C(u)}.$$

Hence, a natural estimator of Λ is given by

$$\widehat{\Lambda}_n(x) = \int_0^x \frac{dF_n^*(u)}{C_n(u)} = \sum_{i=1}^n \frac{I(X_i \leq x)}{nC_n(X_i)},$$

which is the usual so-called Nelson-Aalen estimator of Λ. Moreover, $\widehat{\Lambda}_n$ is the cumulative hazard function of the PL-estimator $\widehat{F}_n$ defined in (3).

We consider the well-known kernel estimator for f as

$$f_n(t) = \frac{1}{h_n} \int_0^\infty K\left(\frac{t - x}{h_n}\right) d\widehat{F}_n(x),$$

where K is a smooth kernel function and h_n is a sequence of positive bandwidths tending to zero. As an estimator for λ, the hazard rate function of the lifetimes, we shall consider

$$\lambda_n(t) = \frac{1}{h_n} \int_0^\infty K\left(\frac{t-x}{h_n}\right) d\widehat{\Lambda}_n(x).$$

For the case in which the lifetime observations are mutually independent, the estimation for density and hazard rate has been studied extensively by many authors during recent years, for example, Uzunoḡullari and Wang (1992), Gijbels and Wang (1993), Sun (1997), Sun and Zhou (1998), and Arcones and Giné (1995). However, for the case that truncated observations are dependent, there are preciously few results available. Under strong mixing condition, Sun and Zhou (2001), established the uniform consistency and asymptotic normality of the f_n and λ_n. Fakoor and Jomhoori (2011) applied the strong Gaussian approximation technique to prove the uniform consistency of kernel density estimators with truncated strong mixing data.

It is well known that the most widely accepted stochastic measure of the global performance of a kernel estimator is its integrated square error, defined by,

$$ISE(f_n) = \int (f_n(t) - f(t))^2 w(t) dt,$$

where w is a nonnegative weighted function. The corresponding deterministic measure of the accuracy of f_n is the mean integrated square error given by

$$MISE(f_n) = \int E\left(f_n(t) - f(t)\right)^2 w(t) dt.$$

Integrated square error is often used in simulation studies to measure the performance of density estimators. It is also used implicitly in adaptive constructions of estimators, when the aim is to minimize mean integrated square error in some sense. Both these applications involve the assumption that integrated square error is somehow close to mean integrated square error. The central limit theorem for ISE provides an explicit description of the order of this closeness, by showing that

$$c(n)\{ISE(f_n) - MISE(f_n)\} \to N(0,1),$$

in distribution as $n \to \infty$, where $c(n), n \geq 1$ is a sequence of positive constants diverging to infinity. The asymptotic behavior of ISE has been

studied extensively by many authors. Bickel and Rosenblatt (1973) employed the uniform strong approximation of the empirical process by the Brownian bridge to obtain a central limit theorem for the ISE of the Rosenblat-Parzen kernel estimators of a density function. Hall (1984) derived central limit theorem for the ISE of density estimator using martingale theory and U-statistics approach. In the right censored case, Yang (1993) employed the martingale techniques by Gill (1983) to get a central limit theorem for the ISE of the product-limit kernel density estimators. Zhang (1996) obtained a simple asymptotic expression for the mean integrated square error of the kernel estimator f_n, and then derived an asymptotically optimal bandwidth for f_n. Zhang (1998) applied the technique of strong approximation to establish an asymptotic expansion for ISE of the kernel density estimate f_n. Sun and Zheng (1999) proved a central limit theorem for the ISE of the kernel hazard rate estimators and also presented an asymptotic representation of the $MISE$ for kernel hazard rate estimators in left truncated and right censored data. For the case that censored observations are dependent, Jomhoori et al. (2007), (2011) studied the central limit theorem for ISE of kernel hazard rate and kernel density estimators.

However, for the case that truncated observations are dependent, there are no results available. The main aim of this paper is to derive a central limit theorem for ISE of kernel density and kernel hazard rate estimators when the truncated data exhibit some kind of dependence. As a corollory we obtain CLT for Hellinger distance.

Among various mixing conditions used in the literature, strong mixing, whose definition is given below is reasonably weak and has many practical applications. Many stochastic processes and time series are known to be strong mixing. In particular, the stationary autoregressive-moving average (ARMA) processes, which are widely applied in time series analysis, are strong mixing with exponential mixing coefficient, i.e., $\alpha(n) = e^{-\nu n}$ for some $\nu > 0$.

Definition 1.1. Let $\{X_i, i \geq 1\}$ denote a sequence of random variables. Given a positive integer m, set

$$\alpha(m) = \sup_{k\geq 1}\{|P(A \cap B) - P(A)P(B)| \ ; \ A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+m}^{\infty}\}, \quad (7)$$

where $\mathcal{F}_i^k$ denote the σ-field of events generated by $\{X_j; i \leq j \leq k\}$. The sequence is said to be strongly mixing (α-mixing) if the mixing coefficient $\alpha(m) \to 0$ as $m \to \infty$.

For such mixing condition, the following basic inequality (see, Kim, 1994) is well known. Let ξ and η be measurable with respect to $\mathcal{F}_1^k$ and $\mathcal{F}_{k+n}^\infty$, respectively. Then, under α-mixing condition,

$$|E(\xi\eta) - E(\xi)E(\eta)| \leq 2\pi[\alpha(n)]^{1-(1/p+1/q)} E^{1/p}|\xi|^p E^{1/q}|\eta|^q, \qquad (8)$$

where $E|\xi|^p, E|\eta|^q < \infty$ for $1 \leq p, q \leq \infty$ with $1/p + 1/q < 1$.

For the sake of simplicity, the assumptions used in this paper are as follows.

Assumptions.

1. Suppose that $\{X_i, i \geq 1\}$ is a sequence of stationary strongly mixing random variables with continuous d.f. F and mixing coefficient α as defined on (1.7).
2. Suppose that $\{T_i, i \geq 1\}$ is a sequence of i.i.d. random variables with continuous d.f. G which are independent of $\{X_i, i \geq 1\}$ and $a_G < a_F$.
3. $\alpha(n) = O(n^{-\nu})$ for some $\nu > 3$ and $b_n = n^{-1/2}(\log n)^{-\delta}$, for some $\delta > 0$ depending on ν.
4. K is a symmetric, bounded, nonnegative and continuously differentiable function on $[-1, 1]$ and satisfies the following conditions:
$$\int_{-1}^{1} K(t)dt = 1, \qquad \int_{-1}^{1} tK(t)dt = 0, \qquad \int_{-1}^{1} t^2K(t)dt \neq 0,$$
5. The weight function w is continuously differentiable which is supported on $[0, \tau]$, where $\tau < b_F$.

2 Main Results

The main results of this paper are the following theorems, which present a central limit theorem for $ISE(f_n)$ and $ISE(\lambda_n)$. The proof is deferred to Section 3.

The integrated square error of f_n on the interval $[0, \tau], \tau < b_F$, is defined by

$$ISE(f_n) = \int_0^\tau (f_n(t) - f(t))^2 w(t)dt. \qquad (9)$$

Let

$$d_{n1} = \int_0^\tau (\bar{f}_n(t) - f(t))^2 w(t)dt,$$

where

$$\bar{f}_n(t) = h_n^{-1} \int_0^\infty K\left(\frac{t-u}{h_n}\right) f(u)du,$$

and

$$\begin{aligned}\sigma_{01}^2 &= \left(\int_{-1}^1 u^2 K(u)du\right)^2 \int (f''(t)w(t))^2 \frac{f(t)}{\bar{G}(t)}dt \\ &+ 16\pi \sum_{i=1}^\infty \alpha^{1/2}(i) \left(\int_{-1}^1 u^2 K(u)du\right)^2 \left(\int (\frac{f''(t)w(t)}{\bar{G}(t)})^4 f(t)dt\right)^{1/2} \quad (10)\end{aligned}$$

Theorem 2.1. *Let h_n be a sequence of positive bandwidths which satisfies $h_n(\log n)^\delta \to \infty$ as $n \to \infty$ for some $\delta > 0$. Under stated assumptions, we have*

$$h_n^{-2}\sqrt{n}(ISE(f_n) - d_{n1}) \xrightarrow{\mathcal{L}} N(0, \sigma_1^2),$$

where $\sigma_1^2 \in (0, \sigma_{01}^2)$.

The integrated square error of λ_n on the interval $[0, \tau]$ is defined by

$$ISE(\lambda_n) = \int_0^\tau (\lambda_n(t) - \lambda(t))^2 w(t)dt. \quad (11)$$

Let

$$d_{n3} = \int_0^\tau (\overline{\lambda}_n(t) - \lambda(t))^2 w(t)dt,$$

where

$$\overline{\lambda}_n(t) = h_n^{-1} \int_0^\infty K\left(\frac{t-u}{h_n}\right) d\Lambda(x),$$

and

$$\begin{aligned}\sigma_{03}^2 &= \left(\int_{-1}^1 u^2 K(u)du\right)^2 \int (\lambda''(t)w(t))^2 \frac{\lambda(t)}{c(t)}dt \\ &+ 16\pi \sum_{i=1}^\infty \alpha^{1/2}(i) \left(\int_{-1}^1 u^2 K(u)du\right)^2 \left(\int (\frac{\lambda''(t)w(t)}{C(t)})^4 \lambda(t)dt\right)^{1/2} \quad (12)\end{aligned}$$

Theorem 2.2. *Let h_n be a sequence of positive bandwidths which satisfies $h_n(\log n)^\delta \to \infty$ as $n \to \infty$ for some $\delta > 0$. Under stated assumptions, we have*

$$h_n^{-2}\sqrt{n}(ISE(\lambda_n) - d_{n3}) \xrightarrow{\mathcal{L}} N(0, \sigma_3^2),$$

where $\sigma_3^2 \in (0, \sigma_{03}^2)$.

Another stochastic measure of accuracy is Hellinger distance. The Hellinger distances between f_n, f also λ_n, λ are defined respectively by

$$HD(f_n) = \int_0^\tau \left(\sqrt{f_n(t)} - \sqrt{f(t)}\right)^2 dt, \tag{13}$$

$$HD(\lambda_n) = \int_0^\tau \left(\sqrt{\lambda_n(t)} - \sqrt{\lambda(t)}\right)^2 dt. \tag{14}$$

Let

$$\begin{aligned} d_{n2} &= \int_0^\tau \frac{(\bar{f}_n(t) - f(t))^2}{4f(t)} dt, \\ \sigma_{02}^2 &= \frac{1}{16}\left(\int_{-1}^1 u^2 K(u)du\right)^2 \int \frac{(f''(t))^2}{f(t)\bar{G}(t)} dt \\ &+ \frac{\pi}{4}\sum_{i=1}^\infty \alpha^{1/2}(i)\left(\int_{-1}^1 u^2 K(u)du\right)^2 \left(\int \frac{(f''(t))^4}{f^3(t)\bar{G}^4(t)} dt\right)^{1/2}. \end{aligned}$$

Corollary 2.1. *Under the stipulated assumptions on the Theorem* 2.1

$$h_n^{-2}\sqrt{n}(HD(f_n) - d_{n2}) \xrightarrow{\mathcal{L}} N(0, \sigma_2^2),$$

where $\sigma_2^2 \in (0, \sigma_{02}^2)$.

Let

$$\begin{aligned} d_{n4} &= \int_0^\tau \frac{(\overline{\lambda}_n(t) - \lambda(t))^2}{4\lambda(t)} dt, \\ \sigma_{04}^2 &= \frac{1}{16}\left(\int_{-1}^1 u^2 K(u)du\right)^2 \int \frac{(\lambda''(t))^2}{\lambda(t)C(t)} dt \\ &+ \frac{\pi}{4}\sum_{i=1}^\infty \alpha^{1/2}(i)\left(\int_{-1}^1 u^2 K(u)du\right)^2 \left(\int \frac{(\lambda''(t))^4}{\lambda^3(t)(C)^4(t)} dt\right)^{1/2}. \end{aligned}$$

Corollary 2.2. *Under the stipulated assumptions on the Theorem* 2.2

$$h_n^{-2}\sqrt{n}(HD(\lambda_n) - d_{n4}) \xrightarrow{\mathcal{L}} N(0, \sigma_4^2),$$

where $\sigma_4^2 \in (0, \sigma_{04}^2)$.

3 Proofs

The proof of Theorem 2.1 and 2.2 is based on the following lemmas. We begin with introducing some further notations. We define

$$Q_{n1} = 2\int_0^\tau (f_n(t) - \bar{f}_n(t))(\bar{f}_n(t) - f(t))w(t)dt, \tag{15}$$

$$Q_{n2} = \int_0^\tau (f_n(t) - \bar{f}_n(t))^2 w(t)dt. \tag{16}$$

$$\overline{\lambda}_n(t) = h_n^{-1}\int_0^\infty K\left(\frac{t-u}{h_n}\right) d\Lambda(u),$$

$$Q_{n3} = 2\int_0^\tau (\lambda_n(t) - \overline{\lambda}_n(t))(\overline{\lambda}_n(t) - \lambda(t))w(t)dt, \tag{17}$$

$$Q_{n4} = \int_0^\tau (\lambda_n(t) - \overline{\lambda}_n(t))^2 w(t)dt. \tag{18}$$

Let

$$\xi(x,t,y) = \frac{I(x \le y)}{C(x)} - \int_0^y \frac{I(t \le u \le x)}{C^2(u)} dF^*(u).$$

It is easy to see that,

$$E(\xi(X_i, T_i, y)) = 0 \quad , \quad Cov(\xi(X_i, T_i, y_1), \xi(X_i, T_i, y_2)) = \int_0^{y_1 \wedge y_2} \frac{dF^*(u)}{C^2(u)}.$$

Lemma 3.1. *Under the assumptions of Theorem* 2.1*, we have*

$$h_n^{-2}\sqrt{n}Q_{n1} \xrightarrow{\mathcal{L}} N(0, \sigma_1^2), \tag{19}$$

where $\sigma_1^2 \in (0, \sigma_{01}^2)$*, and* σ_{01}^2 *is defined in* (10).

Proof. Integrating by parts and Theorem (2.1) of Sun and Zhou (2001) imply

$$\begin{aligned} f_n(t) - \bar{f}_n(t) &= -h_n^{-1} \int (\widehat{F}_n(x) - F(x)) dK\left(\frac{t-x}{h_n}\right) \\ &= (nh_n)^{-1} \sum_{i=1}^{n} \int (1 - F(x)) \xi(X_i, T_i, x) dK\left(\frac{t-x}{h_n}\right) \\ &+ O(h_n^{-1} b_n), \quad a.s. \end{aligned}$$

By using Taylor expansion

$$\bar{f}_n(t) - f(t) = \frac{h_n^2}{2} f''(t) \int_{-1}^{1} u^2 K(u) du + o(h_n^2).$$

Therefore

$$Q_{n1} = \frac{1}{nh_n} \sum_{i=1}^{n} V_{ni} + O(h_n b_n), \quad a.s.,$$

where

$$V_{ni} = 2 \int_0^{\tau} (\bar{f}_n(t) - f(t)) \int (1 - F(x)) \xi(X_i, T_i, x) dK\left(\frac{t-x}{h_n}\right) w(t) dt.$$

It is clear to see that $\{V_{ni}\}$ is a sequence of stationary α-mixing bounded random variables. It can be easily checked that

$$\begin{aligned} E(V_{ni}) &= 0, \\ E(V_{ni}^2) &\leq h_n^6 \gamma \left(\int_{-1}^{1} u^2 K(u) du\right)^2 \int (f''(x) w(x))^2 \frac{f(x)}{G(x)} dx + o(h_n^6). \end{aligned}$$

We may write, $V_{ni} = V_{ni1} - V_{ni2}$, where

$$V_{ni1} = 2 \int_0^{\tau} (\bar{f}_n(t) - f(t)) \int (1 - F(x)) \frac{I(X_i \leq x)}{C(X_i)} dK\left(\frac{t-x}{h_n}\right) w(t) dt,$$

and

$$\begin{aligned} V_{ni2} &= 2 \int_0^{\tau} (\bar{f}_n(t) - f(t)) \int (1 - F(x)) \\ &\times \int_0^{x} \frac{I(T_i \leq u \leq X_i)}{C^2(u)} dF^*(u) dK\left(\frac{t-x}{h_n}\right) w(t) dt. \end{aligned}$$

It can be shown that

$$E|V_{ni1}|^m \leq h_n^{3m} \left|\int_{-1}^{1} u^2 K(u) du\right|^m \int |f''(x) w(x)|^m \frac{f(x)}{G^m(x)} dx + o(h_n^{3m}),$$

$$E|V_{ni2}|^m \leq h_n^{3m}|\int_{-1}^{1} u^2K(u)du|^m \int |f''(x)w(x)|^m \frac{f(x)}{C^m(x)}dx + o(h_n^{3m}).$$

Therefore

$$E^{1/m}|V_{ni}|^m \leq 2h_n^3|\int_{-1}^{1} u^2K(u)du| \left(\int |\frac{f''(x)w(x)}{C(x)}|^m f(x)dx\right)^{1/m} + o(h_n^3).$$

Applying Lemma 2.4 in Kim (1994), with $r = 2$ and $p = q = 4$ we get

$$\begin{aligned} |E(V_{n1}V_{nj})| &\leq 8\pi h_n^6 \alpha^{1/2}(j-1)\left(\int_{-1}^{1} u^2K(u)du\right)^2 \\ &\times \left(\int (\frac{f''(x)w(x)}{C(x)})^4 f(x)dx\right)^{1/2} + o(h_n^6). \end{aligned}$$

So, we have

$$Var(\sum_{i=1}^{n} V_i) = n\sigma_1^2(1 + o(1)),$$

where

$$\begin{aligned} \sigma_1^2 &= E(V_{ni}^2) + 2\sum_{i=1}^{\infty} E(V_{n1}V_{ni}) \\ &\leq h_n^6 \left(\int_{-1}^{1} u^2K(u)du\right)^2 \int (f''(x)w(x))^2 \frac{\gamma f(x)}{G(x)}dx \\ &+ 16\pi h_n^6 \alpha^{1/2}(j-1)\left(\int_{-1}^{1} u^2K(u)du\right)^2 \left(\int (\frac{f''(x)w(x)}{C(x)})^4 f(x)dx\right)^{1/2} \\ &+ o(h_n^6). \end{aligned}$$

Applying Theorem 18.5.4 in Ibragimov and Linnik (1971) we obtain the result.□

Lemma 3.2. *Under the assumptions of Theorem* 2.2*, we have*

$$h_n^{-2}\sqrt{n}Q_{n3} \xrightarrow{\mathcal{L}} N(0, \sigma_3^2), \tag{20}$$

where $\sigma_3^2 \in (0, \sigma_{03}^2)$*, and* σ_{03}^2 *is defined in* (12)*.*

Proof. Applying Theorem (2.1) of Sun and Zhou (2001),we have

$$\begin{aligned} \lambda_n(t) - \overline{\lambda}_n(t) &= -h_n^{-1}\int (\widehat{\Lambda}_n(x) - \Lambda(x))dK\left(\frac{t-x}{h_n}\right) \\ &= (nh_n)^{-1}\sum_{i=1}^{n}\int_0^{\infty} \xi(X_i, T_i, x)dK\left(\frac{t-x}{h_n}\right) + O(h_n^{-1}b_n). \end{aligned}$$

By using Taylor expansion

$$\overline{\lambda}_n(t) - \lambda(t) = \frac{h_n^2}{2}\lambda''(t)\int_{-1}^{1} u^2 K(u)du + o(h_n^2).$$

Therefore

$$Q_{n3} = \frac{1}{nh_n}\sum_{i=1}^{n} U_{ni} + O(h_n^{-1}b_n) \quad a.s.,$$

where

$$U_{ni} = 2\int_0^{\tau}(\overline{\lambda}_n(t) - \lambda(t))\int_0^{\infty}\xi(X_i, T_i, x)dK\left(\frac{t-x}{h_n}\right)w(t)dt.$$

It is clear to see that $\{U_{ni}\}$ is a sequence of stationary α-mixing bounded random variables. It can be easily checked that

$$\begin{aligned} E(U_{ni}) &= 0, \\ E(U_{ni}^2) &\leq h_n^6\left(\int_{-1}^{1} u^2K(u)du\right)^2\int(\lambda''(x)w(x))^2\frac{\lambda(x)}{C(x)}dx + o(h_n^6). \end{aligned}$$

We may write, $U_{ni} = U_{ni1} - U_{ni2}$, where

$$U_{ni1} = 2\int_0^{\tau}(\overline{\lambda}_n(t) - \lambda(t))\int_0^{\infty}\frac{I(X_i \leq x)}{C(X_i)}dK\left(\frac{t-x}{h_n}\right)w(t)dt,$$

and

$$U_{ni2} = 2\int_0^{\tau}(\overline{\lambda}_n(t) - \lambda(t))\int_0^{\infty}\int_0^{x}\frac{I(T_i \leq u \leq X_i)}{C^2(u)}dF^*(u)dK\left(\frac{t-x}{h_n}\right)w(t)dt.$$

$$E|U_{ni1}|^m \leq h_n^{3m}|\int_{-1}^{1} u^2K(u)du|^m\int|\lambda''(x)w(x)|^m\frac{\lambda(x)}{C^m(x)}dx + o(h_n^{3m}),$$

$$E|U_{ni2}|^m \leq h_n^{3m}|\int_{-1}^{1} u^2K(u)du|^m\int|\lambda''(x)w(x)|^m\frac{\lambda(x)}{C^m(x)}dx + o(h_n^{3m}).$$

Therefore

$$E^{1/m}|U_{ni}|^m \leq 2h_n^3|\int_{-1}^{1} u^2K(u)du|\left(\int|\frac{\lambda''(x)w(x)}{C(x)}|^m\lambda(x)dx\right)^{1/m} + o(h_n^3).$$

Applying Lemma 2.4 in Kim (1994), with $r = 2$ and $p = q = 4$ we get

$$\begin{aligned} |E(U_{n1}U_{nj})| &\leq 8\pi h_n^6\alpha^{1/2}(j-1)\left(\int_{-1}^{1} u^2K(u)du\right)^2 \\ &\times \left(\int(\frac{\lambda''(x)w(x)}{C(x)})^4\lambda(x)dx\right)^{1/2} + o(h_n^6). \end{aligned}$$

So, we have

$$Var(\sum_{i=1}^{n} U_i) = n\sigma_3^2(1+o(1)),$$

where

$$\begin{aligned}
\sigma_3^2 &= E(U_{ni}^2) + 2\sum_{i=2}^{\infty} E(U_{n1}U_{ni}) \\
&\leq h_n^6 \left(\int_{-1}^{1} u^2 K(u)du\right)^2 \int (\lambda''(x)w(x))^2 \frac{\lambda(x)}{C(x)} dx \\
&+ 16\pi h_n^6 \alpha^{1/2}(j-1) \left(\int_{-1}^{1} u^2 K(u)du\right)^2 \left(\int (\frac{\lambda''(x)w(x)}{C(x)})^4 \lambda(x)dx\right)^{1/2} \\
&+ o(h_n^6).
\end{aligned}$$

Applying Theorem 18.5.4 in Ibragimov and Linnik (1971), we obtain the result.□

Lemma 3.3. *Under the assumptions of Theorem* 2.1*, we have*

$$h_n^{-2}\sqrt{n}Q_{n2} = o_p(1) \tag{21}$$

and

$$h_n^{-2}\sqrt{n}Q_{n4} = o_p(1). \tag{22}$$

Proof. By applying Corollary (3.3) of Sun and Zhou (2001) , we obtain the results.□

Proof of Theorem 1. By expanding the square in (9), we have

$$ISE(f_n(t)) = Q_{n1} + Q_{n2} + \int_0^{\tau} (\bar{f}_n(t) - f(t))^2 dt,$$

where Q_{n1} and Q_{n2} are defined in (15), (16). Applying Lemma 3.1 and Corollary (3.3) of Sun and Zhou (2001) , we obtain the result. □

Proof of Theorem 2. By expanding the square in (11), we have

$$ISE(\lambda_n(t)) = Q_{n3} + Q_{n4} + \int_0^{\tau} (\overline{\lambda}_n(t) - \lambda(t))^2 dt,$$

where Q_{n3} and Q_{n4} are defined in (17), (18). Applying Lemma 3.2 and Corollary (3.3) of Sun and Zhou (2001), we obtain the result.

□

Proof of Corollary 1. Let

$$\varepsilon_n(t) = \frac{\sqrt{f_n(t)} - \sqrt{f(t)}}{\sqrt{f_n(t)} + \sqrt{f(t)}}.$$

It can be written

$$\begin{aligned}
\sup_{0\le t\le\tau} |\varepsilon_n(t)| &= \sup_{0\le t\le\tau} \frac{|f_n(t) - f(t)|}{(\sqrt{f_n(t)} + \sqrt{f(t)})^2} \\
&\le \sup_{0\le t\le\tau} \frac{|f_n(t) - \bar{f}_n(t)|}{(\sqrt{f_n(t)} + \sqrt{f(t)})^2} + \sup_{0\le t\le\tau} \frac{|\bar{f}_n(t) - f(t)|}{(\sqrt{f_n(t)} + \sqrt{f(t)})^2} \\
&\le \sup_{0\le t\le\tau} \frac{h_n^{-1}|\int_0^\infty (\widehat{F}_n(x) - F(x))dK\left(\frac{t-x}{h_n}\right)}{f(t)} \\
&\quad + \sup_{0\le t\le\tau} \frac{|\bar{f}_n(t) - f(t)|}{f(t)} \\
&\le (\sup_{x\ge 0} |\widehat{\Lambda}_n(x) - \Lambda(x)|) \sup_{0\le t\le\tau} \frac{h_n^{-1}|\int_0^\infty (1 - F(x))dK\left(\frac{t-x}{h_n}\right)}{f(t)} \\
&\quad + O(\frac{\gamma_n^2}{h_n}) + O(h_n^2) \\
&\le O(\frac{\gamma_n}{h_n}) + O(\frac{\gamma_n^2}{h_n}) + O(h_n^2) = o(1) \quad a.s.,
\end{aligned}$$

where $\gamma_n = n^{-1/2}(\log\log n)^{1/2})$. Therefore

$$\begin{aligned}
HD(f_n) &= \int_0^\tau \frac{(f_n(t) - f(t))^2}{4f(t)}dt + \int_0^\tau \frac{\varepsilon_n^2(t) - 2\varepsilon_n(t)}{4f(t)}(f_n(t) - f(t))^2 dt \\
&= \int_0^\tau \frac{(f_n(t) - f(t))^2}{4f(t)}dt + o(1)\int_0^\tau \frac{(f_n(t) - f(t))^2}{4f(t)}dt.
\end{aligned}$$

By applying Theorem 2.1 for $\omega(t) = \frac{1}{4f(t)}$ we obtain the result.□

Proof of Corollary 2. Let

$$\varepsilon_n(t) = \frac{\sqrt{\lambda_n(t)} - \sqrt{\lambda(t)}}{\sqrt{\lambda_n(t)} + \sqrt{\lambda(t)}}.$$

It can be written

$$\sup_{0\le t\le\tau} |\varepsilon_n(t)| = \sup_{0\le t\le\tau} \frac{|\lambda_n(t) - \lambda(t)|}{(\sqrt{\lambda_n(t)} + \sqrt{\lambda(t)})^2}$$

$$
\begin{aligned}
&\leq \sup_{0\leq t\leq\tau} \frac{h_n^{-1}|\int_0^\infty(\widehat{\Lambda}_n(x)-\Lambda(x))dK\left(\frac{t-x}{h_n}\right)}{\lambda(t)} \\
&\quad + \sup_{0\leq t\leq\tau} \frac{|\overline{\lambda}_n(t)-\lambda(t)|}{\lambda(t)} \\
&\leq (\sup_{x\geq 0}|\widehat{\Lambda}_n(x)-\Lambda(x)|) \sup_{0\leq t\leq\tau} \frac{h_n^{-1}|\int_0^\infty dK\left(\frac{t-x}{h_n}\right)}{\lambda(t)} + O(h_n^2) \\
&\leq O(\frac{\gamma_n}{h_n}) + O(h_n^2) = o(1) \quad a.s.,
\end{aligned}
$$

Therefore

$$
\begin{aligned}
HD(\lambda_n) &= \int_0^\tau \frac{(\lambda_n(t)-\lambda(t))^2}{4\lambda(t)}dt + \int_0^\tau \frac{\varepsilon_n^2(t)-2\varepsilon_n(t)}{4\lambda(t)}(\lambda_n(t)-\lambda(t))^2dt \\
&= \int_0^\tau \frac{(\lambda_n(t)-\lambda(t))^2}{4\lambda(t)}dt + o(1)\int_0^\tau \frac{(\lambda_n(t)-\lambda(t))^2}{4\lambda(t)}dt.
\end{aligned}
$$

By applying Theorem 2.2 for $\omega(t) = \frac{1}{4\lambda(t)}$, we obtain the result.□

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