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Stochastic and Dependence Comparisons Between Extreme Order Statistics in the Case of Proportional Reversed Hazard Model

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Abstract. Independent random variables $Y_1, \ldots, Y_n$ belongs to the proportional reversed hazard rate (PRHR) model with proportionality parameters $\lambda_1, ..., \lambda_n$, if $Y_k \sim G^{\lambda_k}(x)$, for $k = 1, ..., n$, where G is an absolutely continuous distribution function. In this paper we compare the smallest order statistics, the sample ranges and the ratios of the smallest and largest order statistics of two sets of independent random variables belonging to PRHR model, in the sense of (reversed) hazard rate order, likelihood ratio order and dispersive order, when the variables in one set have proportionality parameters $\lambda_1, ..., \lambda_n$ and the variables in the other set are independent and identically distributed with common parameter $\overline{\lambda} = \sum_{k=1}^{n} \lambda_k/n$. We also compare the relative degree of dependence between the smallest and the largest order statistics of these samples whit respect to the monotone regression dependence order.

Keywords. Dependence ordering, exponentiated distribution, proportional (reversed) hazard rate model, series system, stochastic comparison.

MSC: 60E15, 62G30, 62H05.

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1 Introduction

Two continuous random variables X and Z, with respective survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, density functions f and g, the reversed hazard rate functions $\widetilde{r}_X = f/F$, $\widetilde{r}_Z = g/G$, respectively satisfy the proportional reversed hazard rate (PRHR) model with the proportionality parameter $\lambda > 0$, if $\widetilde{r}_X(t) = \lambda \widetilde{r}_Z(t)$, which is equivalent to the case $F(t) = G^{\lambda}(t)$. This model proposed by Gupta et al (1998) as a dual of the well–known proportional hazard rate (PHR) model $\overline{F}(t) = \overline{G}^{\lambda}(t)$. The class of distributions of the form $G^{\lambda}(.)$ is also known as the exponentiated class of distributions with baseline distribution function G. In recent years, several standard distributions have been generalized via exponentiation; for instance see Mudholkar and Hutson (1996), Nassar and Eissa (2003), Nadarajah (2006) and Gupta and Kundu (2007).

Order statistics and statistics based on them play an important role in different fields of statistics, especially reliability theory. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics of independent and identically distributed (i.i.d.) random variables $X_1, X_2, ..., X_n$ with common distribution function F. It is well known that, the lifetimes of parallel and series systems are correspond to the $X_{n:n}$ and $X_{1:n}$, respectively. However, in practice, systems are often made up of components whose lifetimes are mutually independent but non-identically distributed. It is of general interest to study the impact of heterogeneity among components on the characteristics of a stochastic system.

There is an extensive literature on stochastic comparisons of order statistics when the observations follow the exponential distribution with different hazard rates. Important contributions in this area have been made by Proschan and Sethuraman (1976), Kochar and Rojo (1996), Dykstra et al. (1997), Khaledi and Kochar (2000), Kochar and Xu (2007) and Mao and Hu (2010) among others. This paper, focuses on stochastic comparisons of the extreme order statistics when underlying random variables follow the PRHR model. More precisely, let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of mutually independent random variables. Assume that for $k \in \{1, \ldots, n\}$, $Y_k \sim G^{\lambda_k}(x)$ and $X_k \sim G^{\overline{\lambda}}(x)$, where $\overline{\lambda} = \sum_{k=1}^{n} \lambda_k / n$. In Section 3 we show that $Y_{1:n}$ is less than $X_{1:n}$ whit respect to the likelihood ratio order and dispersive order. It is also showed that $Y_{n:n} - Y_{1:n}$ is larger than $X_{n:n} - X_{1:n}$, while $Y_{1:n}/Y_{n:n}$ is less than $X_{1:n}/X_{n:n}$, whit respect to the usual stochastic order. Another related topic which has attracted some attention in the literature is the problem of comparing the relative degree of dependence between order statistics. Khaledi and Kochar (2005) and Avérous, et. al (2005) investigated the dependence between (generalized) order statistics in the case of i.i.d. random variables. Dolati et al. (2008) used the right tail increasing order to investigate the dependence between $(X_{1:n}, X_{n:n})$ and $(Y_{1:n}, Y_{n:n})$ in the case of PHR model. Genest et al. (2009) extended their result to the monotone regression dependence order. In Section 4, we will show that similar result holds in the case of PRHR model; that is the pair $(X_{1:n}, X_{n:n})$ is more dependent than the pair $(Y_{1:n}, Y_{n:n})$ whit respect to the monotone regression dependence order. Some notions of stochastic ordering and dependence ordering which are necessary for stating the main results are given in Section 2.

2 Preliminaries

Throughout this paper the term increasing (decreasing) is used for monotone nondecreasing (non–increasing). The notation $=^d$, denote the equality in distribution. We first recall some stochastic orders from Shaked and Shanthikumar (2007) which will be used in the sequel. Let X and Y be univariate random variables with the respective survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, density functions f and g, hazard rate functions r_F and r_G, and reversed hazard rate functions $\widetilde{r}_F$ and $\widetilde{r}_G$, respectively.

Definition 2.1. The random variable X is said to be smaller than the random variable Y in the
(i) usual stochastic order (denoted by $X \prec_{\rm st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for all x;
(ii) hazard rate order (denoted by $X \prec_{\rm hr} Y$) if $r_F(x) \geq r_G(x)$ for all x;
(iii) reverse hazard rate order (denoted by $X \prec_{\rm rh} Y$) if $\widetilde{r}_F(x) \leq \widetilde{r}_G(x)$ for all x;
(iv) likelihood ratio order (denoted by $X \prec_{\rm lr} Y$) if $g(x)/f(x)$ is increasing in x;
(v) dispersive order (denoted by $X \prec_{\rm Disp} Y$) if $F^{-1}(v) - F^{-1}(u) \leq G^{-1}(v) - G^{-1}(u)$, for all $0<u<v<1$.

It is well known that

$$X \prec_{\rm lr} Y \Rightarrow X \prec_{\rm hr(rh)} Y \Rightarrow X \prec_{\rm st} Y.$$

To compare the relative degree of dependence between random variables, we need the notions of dependence ordering including: positive quadrant dependence (PQD) order, right-tail increasing (RTI) order, left-tail decreasing (LTD) order and monotone regression dependence (MRD) order. By analogy with the univariate notion of stochastic dominance, the pair (S_1, T_1) is said to be less dependent than the pair (S_2, T_2), in PQD ordering, denoted by $(S_1, T_1) \prec_{\text{PQD}} (S_2, T_2)$, if and only if,

$$C_1(u, v) \leq C_2(u, v), \quad u, v \in (0, 1),$$

where for $i = 1, 2$, C_i is the copula of (S_i, T_i); i.e., the joint distribution function of the pair $(F_i(S_i), G_i(T_i))$; see, e.g., Nelsen (2006). The stronger orderings RTI, LTD and MRD are defined in terms of conditional distributions and their (right continuous) inverses as follows. We refer to Avérous and Dortet-Bernadet (2000) for details. For $i = 1, 2$, write $H^L_{i[s]}(t) = P(T_i \leq t | S_i \leq s)$, $H^R_{i[s]}(t) = P(T_i \leq t | S_i \geq s)$ and $H_{i[s]}(t) = P(T_i \leq t | S_i = s)$. We denote by $\zeta_{p_i} = F_i^{-1}(p_i)$ the p_i^{th} quantile of the marginal distribution of S_i. Then:

Definition 2.2. The pair (S_2, T_2) is said to be more dependent than the pair (S_1, T_1) in
(i) RTI order, denotes by $(T_1|S_1) \prec_{\text{RTI}} (T_2|S_2)$, if and only if, for all $w \in (0, 1)$,

$$0 < p \leq q < 1 \Rightarrow H^R_{2[\zeta_q]} \circ (H^R_{2[\zeta_p]})^{-1}(w) \leq H^R_{1[\zeta_q]} \circ (H^R_{1[\zeta_p]})^{-1}(w); \quad (1)$$

(ii) LTD order, denotes by $(T_1|S_1) \prec_{\text{LTD}} (T_2|S_2)$, if and only if, for all $w \in (0, 1)$,

$$0 < p \leq q < 1 \Rightarrow H^L_{2[\zeta_q]} \circ (H^L_{2[\zeta_p]})^{-1}(w) \leq H^L_{1[\zeta_q]} \circ (H^L_{1[\zeta_p]})^{-1}(w); \quad (2)$$

(iii) MRD order, denotes by $(T_1|S_1) \prec_{\text{MRD}} (T_2|S_2)$, if and only if, for all $w \in (0, 1)$,

$$0 < p \leq q < 1 \Rightarrow H_{2[\zeta_q]} \circ H^{-1}_{2[\zeta_p]}(w) \leq H_{1[\zeta_q]} \circ H^{-1}_{1[\zeta_p]}(w). \quad (3)$$

The following chain of implications is known from Avérous and Dortet-Bernadet (2000)

$$(T_1|S_1) \prec_{\text{MRD}} (T_2|S_2) \Rightarrow \begin{array}{c} (T_1|S_1) \prec_{\text{LTD}} (T_2|S_2) \\ (T_1|S_1) \prec_{\text{RTI}} (T_2|S_2) \end{array} \Rightarrow (S_1, T_1) \prec_{\text{PQD}} (S_2, T_2),$$

which in turn implies that

$$\kappa(S_1, T_1) \leq \kappa(S_2, T_2), \quad (4)$$

where $\kappa(S, T)$, represents Spearman's ρ, Kendall's τ, Gini's coefficient, or indeed any other copula–based measure of concordance satisfying the axioms of Scarsini (1984).

3 Stochastic Comparisons

First we state and prove the results of comparisons of smallest order statistics. Before giving the main result, we recall that

$$X \prec_{\text{hr}} Y \Rightarrow \psi(X) \succ_{\text{rh}} \psi(Y) \quad \text{and} \quad \text{X} \prec_{\text{rh}} \text{Y} \Rightarrow \psi(\text{X}) \succ_{\text{hr}} \psi(\text{Y}), \tag{5}$$

and

$$X \prec_{\text{lr}} Y \Rightarrow \psi(X) \succ_{\text{lr}} \psi(Y), \tag{6}$$

for all decreasing function ψ. Note also that if $X \prec_{\text{st}} Y$, then

$$X \prec_{\text{Disp}} Y \Rightarrow \psi(X) \succ_{\text{Disp}} \psi(Y), \tag{7}$$

for all decreasing convex function ψ. For detail see Theorems 1.B.41, 1.C.8 and 3.B.10. in Shaked and Shanthikumar (2007).

Proposition 3.1. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of mutually independent random variables. Assume that for $k \in \{1, \ldots, n\}$, $Y_k \sim G^{\lambda_k}(x)$ and $X_k \sim G^{\overline{\lambda}}(x)$ where $\overline{\lambda} = \sum_{k=1}^{n} \lambda_k/n$. Then*
(i) $Y_{1:n} \prec_{\text{lr}} X_{1:n}$;
(ii) $Y_{1:n} \prec_{\text{Disp}} X_{1:n}$ if G is of decreasing hazard rate (DHR).

Proof. Let $H(t) = -\log G(t)$, $t > 0$, be the cumulative reversed hazard rate of G. Denoting by $Y_k^* = H(Y_k)$, we notice that Y_k^* is exponentially distributed with hazard rate λ_k for $k = 1, ..., n$. Similarly, $X_k^* = H(X_k)$, is exponentially distributed with hazard rate $\overline{\lambda}$ for $k = 1, ..., n$. Let also $Y_{1:n}^* \leq ... \leq Y_{n:n}^*$ and $X_{1:n}^* \leq ... \leq X_{n:n}^*$ be the order statistics corresponding to the new sets of variables. It is easy to see that

$$Y_{n:n}^* =^d H(Y_{1:n}), \quad Y_{1:n}^* =^d H(Y_{n:n}) \tag{8}$$

and

$$X_{n:n}^* =^d H(X_{1:n}), \quad X_{1:n}^* =^d H(X_{n:n}). \tag{9}$$

For proof of part (i), it follows from Kochar and Xu (2007), that

$$X_{n:n}^* \prec_{\text{lr}} Y_{n:n}^*,$$

that is,

$$H(X_{1:n}) \prec_{\text{lr}} H(Y_{1:n}).$$

Since $H^{-1}(t) = G^{-1}(e^{-t})$, is decreasing in t, the required result follows from (6). To establish (ii), from Dykstra et al. (1997) , we know that,

$$H(X_{1:n}) = X_{n:n}^* \prec_{\text{Disp}} Y_{n:n}^* = H(Y_{1:n}).$$

The assumption that G has DHR, i.e., G is log concave, implies that $H^{-1}(t) = G^{-1}(e^{-t})$ is a decreasing convex function in t. From this fact and that underlying random variables ordered in the usual stochastic order, the result follows from (7).

Remark 3.1. For $x \geq 0$, the distribution function of $Y_{1:n}$ is given by

$$F_{Y_{1:n}}(x) = 1 - \prod_{k=1}^{n} \left(1 - G^{\lambda_k}(x)\right), \tag{10}$$

with the density function

$$f_{Y_{1:n}}(x) = \prod_{k=1}^{n} \left(1 - G^{\lambda_k}(x)\right) \sum_{k=1}^{n} \frac{\lambda_k g(x) G^{\lambda_k - 1}(x)}{1 - G^{\lambda_k}(x)}, \tag{11}$$

where g is the density function of G. Similarly, the distribution function and the density function of $X_{1:n}$ are

$$F_{X_{1:n}}(x) = 1 - \{1 - G^{\overline{\lambda}}(x)\}^n, \tag{12}$$

and

$$f_{X_{1:n}}(x) = n\overline{\lambda} g(x) G^{\overline{\lambda}-1}(x)\{1 - G^{\overline{\lambda}}(x)\}^{n-1}, \tag{13}$$

respectively. Notice that the distribution function of $X_{1:n}$, belongs to the general family of distributions defined by

$$F(x) = 1 - \{1 - G^a(x)\}^b, \quad x > 0, a, b > 0,$$

which is known as the Kumaraswamy's distribution; see, e.g., Cordeiro and de Castro (2010).

Using the results of Proposition 3.1 and above remark, we have the following bounds on the (reversed) hazard rate function and variance of $Y_{1:n}$ in terms of that of $X_{1:n}$.

Corollary 3.2. *Under the conditions of Proposition 3.1,*
(i) the hazard rate $r_{Y_{1:n}}$ and the reversed hazard rate $\widetilde{r}_{Y_{1:n}}$ of $Y_{1:n}$ satisfy

$$r_{Y_{1:n}}(x) \geq \frac{n g(x) \overline{\lambda} G^{\overline{\lambda}-1}(x)}{1 - G^{\overline{\lambda}}(x)}$$

and

$$\widetilde{r}_{Y_{1:n}}(x) \leq \frac{n g(x) \overline{\lambda} G^{\overline{\lambda}-1}(x)\{1 - G^{\overline{\lambda}}(x)\}^{n-1}}{1 - \{1 - G^{\overline{\lambda}}(x)\}^n};$$

(ii) $\mathrm{Var}(Y_{1:n}) \le \mathrm{Var}\{G^{-1}(1-U)^{\frac{1}{\lambda}}\}$, *where* $U \sim F(x) = x^n$, $0 < x < 1$.

Under the weaker condition that if $Y_1^*, \ldots, Y_n^*$ are independent exponential random variables with Y_k^* having hazard rate λ_k, $k = 1, ..., n$ and $Z_1, ..., Z_n^*$ is a random sample of size n from an exponential distribution with common hazard rate $\widetilde{\lambda} = (\prod_{k=1}^n \lambda_k)^{\frac{1}{n}}$, the geometric mean of λ_i's, Khaledi and Kochar (2000) proved that

$$Z_{n:n}^* \prec_{\mathrm{hr}} Y_{n:n}^* \quad \text{and} \quad Z_{n:n}^* \prec_{\mathrm{Disp}} Y_{n:n}^*. \tag{14}$$

The following proposition which follows from (14) to gather with (5) and (6), gives the corresponding result for the PRHR model.

Proposition 3.3. *Let $Z_1, \ldots, Z_n$ and $Y_1, \ldots, Y_n$ be two sets of mutually independent random variables. Assume that for $k \in \{1, \ldots, n\}$, $Y_k \sim G^{\lambda_k}(x)$ and $Z_k \sim G^{\widetilde{\lambda}}(x)$ where $\widetilde{\lambda} = (\prod_{k=1}^n \lambda_k)^{\frac{1}{n}}$. Then*
(i) $Y_{1:n} \prec_{\mathrm{rh}} Z_{1:n}$;
(ii) $Y_{1:n} \prec_{\mathrm{Disp}} Z_{1:n}$ if G is of decreasing hazard rate (DHR).

The following example shows that above result can not be strengthened from the reversed hazard rate order to the likelihood ratio order.

Example 3.1. Let $Y_1, \ldots, Y_n$ be independent random variables with $Y_k \sim F_k(x) = x^{\lambda_k}$, $0 < x < 1$, $k = 1, ..., n$, and let $Z_1, ..., Z_n$ be a random sample from $F(x) = x^{\widetilde{\lambda}}$, $0 < x < 1$. Then using (10)–(13), the hazard rate of $Y_{1:n}$ and $Z_{1:n}$ are given by

$$r_{Y_{1:n}}(x) = \frac{1}{x} \sum_{k=1}^{n} \frac{\lambda_k x^{\lambda_k}}{1 - x^{\lambda_k}},$$

and

$$r_{Z_{1:n}}(x) = \frac{1}{x} \frac{n\widetilde{\lambda} x^{\widetilde{\lambda}}}{1 - x^{\widetilde{\lambda}}}.$$

Let $n = 3$, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 3$, then

$$r_{Y_{1:n}}(0.5) \approx 4.857 < r_{Z_{1:n}}(0.5) \simeq 5.038;$$

i.e., $Y_{1:n}$ is not less than $Z_{1:n}$ in hazard rate order and hence in likelihood ratio order.

Another interesting problem is the comparison of the sample ranges. This topic followed by Kochar and Rajo (1996) where they showed that if $Y_1^*, \ldots, Y_n^*$ are independent exponential random variables with Y_k^* having hazard rate λ_k, $k = 1, ..., n$ and $X_1^*, \ldots, X_n^*$ is a random sample of size n from an exponential distribution with common hazard rate $\overline{\lambda}$, then

$$X_{n:n}^* - X_{1:n}^* \prec_{\rm st} Y_{n:n}^* - Y_{1:n}^*. \tag{15}$$

Kochar and Xu (2007) strengthened this result to

$$X_{n:n}^* - X_{1:n}^* \prec_{\rm rh} Y_{n:n}^* - Y_{1:n}^*. \tag{16}$$

Recently, Genest et al. (2009) extended above result to the likelihood ratio order,

$$X_{n:n}^* - X_{1:n}^* \prec_{\rm lr} Y_{n:n}^* - Y_{1:n}^*. \tag{17}$$

The following proposition provides a result for the sample ranges and sample ratios of the PRHR model.

Proposition 3.4. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of mutually independent random variables. Assume that for $k \in \{1, \ldots, n\}$, $Y_k \sim G^{\lambda_k}(x)$ and $X_k \sim G^{\overline{\lambda}}(x)$ where $\overline{\lambda} = \sum_{k=1}^n \lambda_k/n$. Then*
(i) $X_{n:n} - X_{1:n} \prec_{\rm st} Y_{n:n} - Y_{1:n}$;
(ii) $\frac{Y_{1:n}}{Y_{n:n}} \prec_{\rm st} \frac{X_{1:n}}{X_{n:n}}$,

Proof. Denote by $R_Y = Y_{n:n} - Y_{1:n}$ and $R_X = X_{n:n} - X_{1:n}$ and let $R_Y^* = Y_{n:n}^* - Y_{1:n}^*$ and $R_X^* = X_{n:n}^* - X_{1:n}^*$, be the sample ranges of the heterogenous and homogenous exponential random variables Y_i^*'s and X_i^*'s, respectively. Using the transformations (8), we know that

$$-\log\frac{G(Y_{1:n})}{G(Y_{n:n})} =^d R_Y^*, \quad \text{and} -\log\frac{G(X_{1:n})}{G(X_{n:n})} =^d R_X^*. \tag{18}$$

Since for the independent exponential random variables, the sample range and the smallest order statistics are independent, then one would get

$$\begin{aligned}
\overline{F}_{R_Y}(x) &= P(Y_{n:n} - Y_{1:n} > x) \\
&= \int_x^\infty P(Y_{1:n} \leq y - x | Y_{n:n} = y) f_{Y_{n:n}}(y) dy \\
&= \int_x^\infty P\left(R_{Y^*} \geq -\log\frac{G(y-x)}{G(y)} | Y_{1:n}^* = -\log G(y)\right) f_{Y_{n:n}}(y) dy \\
&= \int_0^\infty \overline{F}_{R_Y^*}\left(\log\frac{G(x+t)}{G(t)}\right) f_{Y_{n:n}}(x+t) dt, \qquad (19)
\end{aligned}$$

where
$$f_{Y_{n:n}}(y) = n\overline{\lambda} g(y) G^{\overline{n\lambda}-1}(y), \quad y > 0.$$
Similarly, the survival function of R_X, for $x > 0$ is,
$$\overline{F}_{R_X}(x) = \int_0^\infty \overline{F}_{R_X^*}\left(\log\frac{G(x+t)}{G(t)}\right) f_{X_{n:n}}(x+t)dt. \tag{20}$$

From (15) we have that $\overline{F}_{R_X^*}(x) \le \overline{F}_{R_Y^*}(x)$ all $x > 0$. Since $f_{Y_{n:n}}(y) = f_{X_{n:n}}(y)$, the expressions (19) and (20) give the required result $\overline{F}_{R_X}(x) \le \overline{F}_{R_Y}(x)$, which proves part (i). Similarly, part (ii) follows from (15) and that the survival functions of $W_Y = Y_{1:n}/Y_{n:n}$ and $W_X = X_{1:n}/X_{n:n}$, are given by
$$\overline{F}_{W_Y}(x) = \int_0^\infty F_{R_Y^*}\left(\log\frac{G(y)}{G(xy)}\right) f_{Y_{n:n}}(y)dy,$$
and
$$\overline{F}_{W_X}(x) = \int_0^\infty F_{R_X^*}\left(\log\frac{G(y)}{G(xy)}\right) f_{Y_{n:n}}(y)dy,$$
respectively, for $0 < x < 1$.

Remark 3.2. At this point, whether the statements of the Proposition 3.4 could be established for (reversed) hazard rate order, likelihood ratio order or dispersive order remain unsolved.

4 Main Results on Dependence Comparisons

Given a random sample $X_1, ..., X_n$, it is easy to see that the smallest and the largest order statistics satisfy the relation
$$P^{\frac{1}{n}}(X_{1:n} > t) + P^{\frac{1}{n}}(X_{n:n} \le t) = 1 \quad \text{for all} \quad t.$$
This observation illustrates a form of dependence between the extreme order statistics $X_{1:n}$ and $X_{n:n}$. Some general results on the dependence structure of order statistics of a random sample could be found in Boland et al. (1996), Schmitz (2004), Khaledi & Kochar (2005) and Avérous et al. (2005). However when the system is consists of independent components whose distributions are different, then this dependence structure is not tractable. A suitable approach for studying such a system is to compare it with a simpler one having i.i.d. components. Dolati et al. (2008) proved that
$$(Y^*_{n:n}|Y^*_{1:n}) \prec_{\text{RTI}} (X^*_{n:n}|X^*_{1:n}). \tag{21}$$

where $Y_1^*, ..., Y_n^*$ are independent random variables with Y_k^* following PHR model $\overline{G}^{\lambda_k}(.)$, $k = 1, ..., n$ and $X_1^*, ..., X_n^*$ is a random sample from $\overline{G}^{\overline{\lambda}}(.)$, $\overline{\lambda} = \sum_{k=1}^{n} \lambda_k/n$. In this section, we consider the above topic in the case of PRHR model. Before giving the result for PRHR model, we recall the relation between RTI and LTD order from Colangelo, et al. (2006). For two pairs (T_1, S_1) and (T_2, S_2) we have that

$$(T_1|S_1) \prec_{\text{RTI}} (T_2|S_2) \Rightarrow (\psi(T_1)|\psi(S_1)) \prec_{\text{LTD}} (\psi(T_2)|\psi(S_2)), \qquad (22)$$

for all decreasing function ψ.

In view of the decreasing transformations defined by (8), it follows that for the PRHR model the RTI order changes to LTD order. The following result provides an equivalent characterization of the LTD order and dispersive ordering of smallest order statistics.

Proposition 4.1. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of mutually independent random variables. Assume that for $k \in \{1, \ldots, n\}$, $Y_k \sim G^{\lambda_k}(.)$ and $X_k \sim G^{\lambda}(.)$, where $\overline{\lambda} = \sum_{k=1}^{n} \lambda_k/n$, and G is a continuous distribution function of DHR. Then the followings are equivalent*
(i) $(Y_{1:n}|Y_{n:n}) \prec_{\text{LTD}} (X_{1:n}|X_{n:n})$;
(ii) $Y_{1:n} \prec_{\text{Disp}} X_{1:n}$.

Proof. Using the transformed variables defined by (8) for $y > x \geq 0$, the conditional distribution of $Y_{1:n}$ given $Y_{n:n} \leq y$, is

$$\begin{aligned}
H_{1[y]}^{L}(x) &= P(Y_{1:n} \leq x | Y_{n:n} \leq y) \\
&= \frac{P(Y_{n:n} \leq y) - P(x \leq Y_k \leq y, k = 1, ..., n)}{P(Y_k \leq y, k = 1, ..., n)} \\
&= 1 - \frac{P(-\log G(y) \leq Y_k^* \leq -\log G(x), k = 1, ..., n)}{P(Y_k^* \geq -\log G(y), k = 1, ..., n)} \\
&= 1 - \prod_{k=1}^{n} \left[1 - \left(\frac{G(x)}{G(y)} \right)^{\lambda_k} \right] \\
&= \overline{F}_{Y_{n:n}^*} \left(\log \frac{G(y)}{G(x)} \right),
\end{aligned}$$

where

$$\overline{F}_{Y_{n:n}^*}(x) = 1 - \prod_{k=1}^{n} \{1 - e^{\lambda_k x}\},$$

is the survival function of $Y_{n:n}^*$. If ζ_p is the p^{th} quantile of the common distribution of $X_{n:n}$ and $Y_{n:n}$, then for $0 < p \le q < 1$ and $0 < u < 1$,

$$H^L_{1[\zeta_q]} \circ (H^L_{1[\zeta_p]})^{-1}(u) = \overline{F}_{Y^*_{n:n}} \left(\overline{F}^{-1}_{Y^*_{n:n}}(u) + \log \frac{G(\zeta_q)}{G(\zeta_p)} \right).$$

Similarly, for $H^L_{2[y]}(x)$, the conditional distribution of $X_{1:n}$ given $X_{n:n} \le y$, we have

$$H^L_{2[\zeta_q]} \circ (H^L_{2[\zeta_p]})^{-1}(u) = \overline{F}_{X^*_{n:n}} \left(\overline{F}^{-1}_{X^*_{n:n}}(u) + \log \frac{G(\zeta_q)}{G(\zeta_p)} \right).$$

According to the definition, $(Y_{1:n}|Y_{n:n}) \prec_{\text{LTD}} (X_{1:n}|X_{n:n})$ holds, if and only if,

$$\overline{F}_{X^*_{n:n}} \left(\overline{F}^{-1}_{X^*_{n:n}}(u) + c \right) \le \overline{F}_{Y^*_{n:n}} \left(\overline{F}^{-1}_{Y^*_{n:n}}(u) + c \right), \tag{23}$$

for all $0 < u < 1$ and $c > 0$; that is

$$X^*_{n:n} \prec_{\text{Disp}} Y^*_{n:n};$$

see, inequality 3.B.4 in Shaked and Shanthikumar (2007). However, under the assumption that G is of DHR, from Proposition 3.1 (ii), the above condition holds if and only if, $Y_{1:n} \prec_{\text{Disp}} X_{1:n}$. This completes the proof.

The flowing result shows that the LTD order between the smallest and the largest order statistics may be generalized to the MRD order.

Proposition 4.2. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of mutually independent random variables. Assume that for $k \in \{1, \ldots, n\}$, $Y_k \sim G^{\lambda_k}(.)$ and $X_k \sim G^{\lambda}(.)$, where $\overline{\lambda} = \sum_{k=1}^n \lambda_k / n$. Then*

$$(Y_{1:n}|Y_{n:n}) \prec_{\text{MRD}} (X_{1:n}|X_{n:n}). \tag{24}$$

Proof. For $y > x \ge 0$, by the same argument as before, the conditional distribution of $Y_{1:n}$ given $Y_{n:n} = y$, is given by

$$H_{1[y]}(x) \;=\; P(Y_{1:n} \le x | Y_{n:n} = y) = \overline{F}_{R^*_Y} \left(\log \frac{G(y)}{G(x)} \right).$$

Let ζ_p denote the p^{th} quantile of the common distribution of $X_{n:n}$ and $Y_{n:n}$, i.e., $G^{n\overline{\lambda}}(x)$. Therefore, for $0 < u < 1$,

$$H^{-1}_{1[\zeta_p]}(u) = G^{-1} \left(G(\zeta_p) e^{-\overline{F}^{-1}_{R^*_Y}(u)} \right),$$

and for $0 < p \leq q < 1$,

$$H_{1[\zeta_q]} \circ H^{-1}_{1[\zeta_p]}(u) = \overline{F}_{R^*_Y}\left(\overline{F}^{-1}_{R^*_Y}(u) + \log\frac{G(\zeta_q)}{G(\zeta_p)}\right). \tag{25}$$

Similarly, for $H_{2[y]}(x)$, the conditional distribution of $X_{1:n}$ given $X_{n:n} = y$, we have

$$H_{2[\zeta_q]} \circ H^{-1}_{2[\zeta_p]}(u) = \overline{F}_{R^*_X}\left(\overline{F}^{-1}_{R^*_X}(u) + \log\frac{G(\zeta_q)}{G(\zeta_p)}\right). \tag{26}$$

Since $c = \log\frac{G(\zeta_q)}{G(\zeta_p)} > 0$, for all $0 < p \leq q < 1$, according to the definition, one can see that (24) holds if and only if

$$\overline{F}_{R^*_X}\left(\overline{F}^{-1}_{R^*_X}(u) + c\right) \leq \overline{F}_{R^*_Y}\left(\overline{F}^{-1}_{R^*_Y}(u) + c\right), \tag{27}$$

for all $0 < u < 1$ and $c > 0$. However, in view of condition 3.B.4 in Shaked and Shanthikumar (2007), the inequality (27) amounts to the statement that

$$R^*_X \preceq_{\text{Disp}} R^*_Y, \tag{28}$$

a fact that was proved in Genest et al. (2009). This completes the proof.

It is of interest to quantify the amount of dependence in the pair $(Y_{1:n}, Y_{n:n})$. Note that the pair $(Y_{1:n}, Y_{n:n})$ is monotone regression dependent, i.e., it is more regression dependence than any pair (S, T) of independent continuous random variables. To this end, using (25) one must show that

$$H_{1[\zeta_q]} \circ H^{-1}_{1[\zeta_p]}(u) \leq u,$$

for all $0<p\leq q<1$ and $u \in (0, 1)$ or equivalently,

$$\overline{F}_{R^*_Y}\left(\overline{F}^{-1}_{R^*_Y}(u) + \log\frac{G(\zeta_q)}{G(\zeta_p)}\right) \leq u,$$

which is an immediate the fact that $\zeta_p \leq \zeta_q$. In the light of Theorem 3.4 of Boland et al. (1996), it follows that $\kappa(Y_{1:n}Y_{n:n}) \geq 0$, for any concordance measure and any heterogenous sample of observations. But the expression for the measures of concordance such as Kendall's τ and Spearman's ρ associated with the pair $(Y_{1:n}, Y_{n:n})$, is not algebraically closed, in view of the heterogeneity. For instance using (4), as a result of Proposition 4.2, one would get

$$\tau(Y_{1:n}, Y_{n:n}) \leq \frac{1}{2n-1},$$

as per Theorem 5 in Schmitz (2004).

The following result shows that the degree of dependence between extreme order statistics of a PRHR model is equal to that of a PHR model.

Proposition 4.3. *Let $Y_1, \ldots, Y_n$ and $V_1, ..., V_n$, be two sets of mutually independent random variables, with $Y_k \sim G^{\lambda_k}(.)$ and $V_k \sim \overline{G}^{\lambda_k}(.)$, $k = 1, ..., n$, where G is a continuous distribution function and $\bar{G} = 1 - G$. Then for any copula–based measure of concordance κ, satisfying the axioms of Scarsini (1984), we have*

$$\kappa(Y_{1:n}, Y_{n:n}) = \kappa(V_{1:n}, V_{n:n}).$$

Proof. Let $H'(t) = -\log\overline{G}(t)$, $t > 0$, be the cumulative hazard rate of G. Denoting by $V_k^* = H'(V_k)$, we notice that V_k^* is exponentially distributed with hazard rate λ_k for $k = 1, ..., n$. It is easy to see that

$$H'(V_{1:n}) =^d V_{1:n}^* \quad \text{and} \quad H'(V_{n:n}) =^d V_{n:n}^*. \tag{29}$$

Recall from (8) that, under the transformation $H(t) = -\log G(t)$,

$$H(Y_{n:n}) =^d V_{1:n}^* \quad \text{and} \quad H(Y_{1:n}) =^d V_{n:n}^*.$$

Now, since $H'(t)$ is an increasing function and $H(t)$ is a decreasing function in t, from Theorem 2.2.4 of Nelsen (2006), the pairs $(V_{1:n}, V_{n:n})$ and $(V_{1:n}^*, Y_{n:n}^*)$ have the same copula structure, while the copula of $(Y_{1:n}, Y_{n:n})$ coincides with the copula of $(-V_{1:n}^*, -V_{n:n}^*)$. The desired result follows from this fact that for a copula–based measure of concordance κ, satisfying Scarsini's axioms and any pair of continuous random variables (T, S), one has $\kappa(T, S) = \kappa(-T, -S)$.

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